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Cosmology Lunch (14 Feb) - Fast directional spherical wavelets for cosmology

0. Intro

- Wavelets powerful signal analysis tool
- Provides both scale and spatial localisation of signal characteristics
- Small sections of sky may be approx. by flat patches, where usual planar Euclidean wavelet analysis may be performed.
- However, to consider full sky maps defined on the sphere we must extend wavelet analysis to spherical geometry.
- Overview:
  - CSWT of Antoine & Vandergheynst
  - Fast directional implementation
  - Cosmological applications (eg. Gaussianity tests)

1. Continuous spherical wavelet transform (CSWT)

Follow formulation of A & V, developed from Group theoretic principles, however possible to understand in terms of simple affine transformations and norm preserving factors.

1.1 Motions & dilations on the sphere

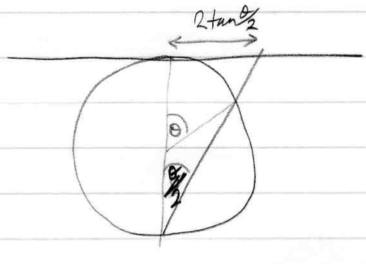
Rotation

$$(R_\rho f)(\omega) = f(\rho^{-1}\omega) \qquad \rho \in SO(3) \qquad f \in L^2(S^2)$$

ρ parameterised by Euler angles (α, β, γ)
Inherently directional $(\alpha, \beta) \to$ position $\gamma \to$ orientation

Dilation

Project sphere to plane (stereographic projection)
→ perform usual Euclidean dilation in plane
→ reproject back onto sphere



$$(D_a f) = f_a(\omega) = \sqrt{\lambda(a,\theta)}\, f(\omega_{1/a}) \qquad a \in \mathbb{R}_*^+$$

where $\omega_a = (\theta_a, \phi)$ and $\tan\frac{\theta_a}{2} = a \tan\frac{\theta}{2}$

cocycle $\lambda(a,\theta) = \frac{4a^2}{[(a^2-1)\cos\theta + (a^2+1)]^2}$ applied to preserve 2-norm.

1.2 Wavelet basis

Wavelet basis constructed from rotations and dilations of an admissible mother spherical wavelet. $\psi \in L^2(S^2)$.

Mother wavelets simply constructed by inverse stereographic projection of admissible planar wavelet

$$\psi_{S^2}(\theta, \phi) = \frac{2}{1+\cos\theta}\, \psi_{\mathbb{R}^2}(r, \phi) \quad \text{where} \quad r = 2\tan \theta/2.$$

So directional mother spherical wavelets simply constructed from directional mother planar wavelets. [OHP: mother wavelets] these wavelets used in NG analysis.

Wavelet basis

$$\{ \psi_{a,\rho} \equiv R_\rho D_a \psi, \quad \rho \in SO(3), \ a \in \mathbb{R}_*^+ \}$$

(Provides an overcomplete set of functions in $L^2(S^2)$.)

1.3 Wavelet transform

Wavelet coefficient given by projection onto each wavelet basis function

Thus for $s \in L^2(S^2)$

$$W(a,\rho) = \int_{S^2} (R_\rho \psi_a)^*(\omega)\, s(\omega)\, d\mu(\omega) \quad \text{where} \quad d\mu(\omega) = \sin\theta\, d\theta\, d\phi$$

(usual rotationally invariant measure on sphere)

Directional in nature, however important to note that only local orientations make any sense on S^2.

2. Fast directional CSWT implementation

(Based on fast spherical convolution of Wandelt & Gorski).

Spherical harmonic representation

$$W(\rho) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \left[D^l_{mm'}(\rho)\, \psi_{lm} \right]^* s_{lm}$$

Rotations expressed using Wigner rotation matrices $D^l_{mm'}(\rho)$ which may be decomposed as

$$D^l_{mm'}(\alpha, \beta, \gamma) = e^{-im\alpha}\, d^l_{mm'}(\beta)\, e^{-im\gamma}$$

Exploit this relationship by factorising rotation into two separate rotations which contain constant $\pm \pi/2$ polar rotations.

$$R_{\alpha,\beta,\gamma} = R_{\alpha-\pi/2,\, -\pi/2,\, \beta}\; R_{0,\, \pi/2,\, \gamma+\pi/2}$$

Then all rotations appear in argument of complex exponentials. Rearrange to a form where 3 of the 4 summations may be performed simultaneously using FFT.

[OHP: Show overhead of complexity and typical execution times.]

Analysis not feasible without fast algorithm.

3. Applications (overview)

Any area where wavelet analysis of full sky maps required (ie. when scale & spatial localisation required)

Examples:

- Gaussianity of CMB anisotropies (CSWT linear ∴ Gaussian sky ⇒ Gaussian coeff.)
- Compact object detection (directional sources)
- CMB-LSS cross-correlation in wavelet space
- others ...

4. Non-Gaussianity in WMAP 1-year data (astroph/0406604)

- CSWT linear
  ∴ Gaussian sky ⇒ Gaussian coeff
- Ability to probe different scales, positions and orientations
  Important to ensure non-Gaussian sources present at certain scales say, are not concealed by predominant Gaussianity of other scales.
- Look for deviations in skewness & kurtosis of wavelet coeff.
- 1000 Monte Carlo simulations to provide significance meas.

[Figure 6.]
Describe figure and results

Consider most significant detection made w/ each wavelet in more detail
[Figure 7]

Skewness & kurtosis dist^n s not necessary Gaussian
∴ Cannot infer significance from No
Perform aditional significance tests
[Table 2]
Treating each wavelet separately, we search through the Gaussian simulations to determine the number of maps that have an equivalent or greater deviation in any of the test statistics
- either skewness or kurtosis [- calculated from that map using the given wavelet.]
→ most conservative means of constructing significance levels.

Localised regions: skewness/kurtosis flagged
[Figure 9]
Describe technique (thresholding; which wavelet each corresponds to, etc.)
Cross-correlations between regions
Regions removed (detections removed for most significant cases, reduced for other cases.)
∴ Detected deviation regions would indeed appear to be source of non-Gaussianity
(Origin: not atypical noise dispersion; secondary; systematics or CMB?)

Don't cover χ^2 test or noise analysis unless requested.