Full-sky interferometry

Simulating full-sky interferometric observations

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http://arxiv.org/abs/0803.2165

Motivation (and disclaimer)

- Aperture array interferometers, such as that proposed for the SKA, will see a large portion of the sky.
- Usual Fourier transform approach for simulating visibilities relies on a tangent plane approximation that is only valid for small fields of view.
- We address the forward wide field imaging problem and consider full-sky contributions to the visibilities observed by an interferometer, ensuring that contamination due to wide sidelobes of the primary beam is not neglected.
- Disclaimer
- Outline of talk
  - Harmonic analysis
  - Coordinate systems
  - Computing visibilities (and image reconstruction)
  - Preliminary simulations
  - Fast wavelet methods

Spherical harmonics

- A square integrable function on the sphere $F \in \mathrm{L}^2(\mathrm{S}^2,\ \mathrm{d}\Omega)$ may be represented by the spherical harmonic expansion

$$F(\hat{\mathbf{s}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} F_{\ell m} Y_{\ell m}(\hat{\mathbf{s}}) \ .$$

- The spherical harmonic coefficients are given by the usual projection onto the spherical harmonic basis functions:

$$F_{\ell m} = \int_{\mathrm{S}^2} F(\hat{\mathbf{s}})\ Y_{\ell m}^*(\hat{\mathbf{s}})\ \mathrm{d}\Omega(\hat{\mathbf{s}}) \ ,$$

where $\mathrm{d}\Omega(\hat{\mathbf{s}}) = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the usual rotation invariant measure on the sphere and $\hat{\mathbf{s}} = (\theta, \varphi) \in \mathrm{S}^2$ denote spherical coordinates with colatitude $\theta \in [0, \pi]$ and longitude $\varphi \in [0, 2\pi)$.

- Useful properties and relations
  - Orthogonality

$$\int_{\mathrm{S}^2} Y_{\ell m}(\hat{\mathbf{s}})\ Y_{\ell' m'}^*(\hat{\mathbf{s}})\ \mathrm{d}\Omega(\hat{\mathbf{s}}) = \delta_{\ell\ell'}\delta_{mm'}$$

  - Addition theorem

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{s}}) Y_{\ell m}^*(\hat{\mathbf{s}}') = \frac{2\ell+1}{4\pi} P_\ell(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}')$$

  - Jacobi-Anger expansion

$$\mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{y}} = \sum_{\ell=0}^{\infty} (2\ell+1)\mathrm{i}^\ell j_\ell(\|\boldsymbol{x}\|\|\boldsymbol{y}\|) P_\ell(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}})$$

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Rotations

- Rotations on the sphere $\mathcal{R}$ characterised by the the rotation group $\mathrm{SO}(3)$, which we parameterise in terms of the three Euler angles $\rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3)$, where $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi)$.
- Rotation of coordinate vector performed by multiplication with 3×3 rotation matrix

$$\mathbf{R}(\rho) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma) \ ,$$

  where $\mathbf{R}_z(\vartheta)$ and $\mathbf{R}_y(\vartheta)$ are rotation matrices representing rotations by ϑ about the z and y axis respectively (adopt zyz Euler convention).
- Rotation of function on the sphere defined by

$$\big(\mathcal{R}(\rho)F\big)(\hat{\mathbf{s}}) = F\big(\mathbf{R}^{-1}(\rho)\hat{\mathbf{s}}\big) \ .$$

- Rotation of function on sphere may be performed more generally (*i.e.* pixelisation independent) and accurately through harmonic space representation. Harmonic coefficients of a rotated function are related to the coefficients of the original function by

$$\big(\mathcal{R}(\rho)F\big)_{\ell m} = \sum_{n=-\ell}^{\ell} D^{\ell}_{mn}(\rho)\, F_{\ell n} \ ,$$

  where the Wigner D-functions $D^{\ell}_{mn}(\rho)$ provide the irreducible unitary representation of the rotation group $\mathrm{SO}(3)$.
  - For computational purposes, the Wigner functions may be decomposed as $D^{\ell}_{mn}(\alpha, \beta, \gamma) = \mathrm{e}^{-\mathrm{i}m\alpha}\, d^{\ell}_{mn}(\beta)\, \mathrm{e}^{-\mathrm{i}n\gamma}$; $d^{\ell}_{mn}(\beta)$ may then be computed rapidly using recursion formulae (Risbo [6]).

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Coordinate systems

- The complex visibility measured by an interferometer is given by the coordinate free definition

$$\mathcal{V}(\boldsymbol{u}) = \int_{\mathrm{S}^2} A(\boldsymbol{\sigma}) I(\boldsymbol{\sigma}) \mathrm{e}^{-\mathrm{i}2\pi \boldsymbol{u}\cdot\boldsymbol{\sigma}} \, \mathrm{d}\Omega \, .$$

- In this coordinate free definition, $\boldsymbol{\sigma}$ is the representation of $\hat{\boldsymbol{s}}$ in a coordinate system centred on $\hat{\boldsymbol{s}}_0$. The translation $\boldsymbol{\sigma} = \hat{\boldsymbol{s}} - \hat{\boldsymbol{s}}_0$ represents the transformation between the global coordinate frame of $\hat{\boldsymbol{s}}$ and the local coordinate frame of $\boldsymbol{\sigma}$.
- In general, one can transform vectors between global coordinates and local coordinates relative to $\hat{\boldsymbol{s}}_0$, through a rotation by $\hat{\boldsymbol{s}}_0$.
- The rotation $\mathcal{R}_0 \equiv \mathcal{R}(\varphi_0, \theta_0, 0)$, where (θ_0, φ_0) are the spherical coordinates of $\hat{\boldsymbol{s}}_0$, transforms the local coordinate frame relative to $\hat{\boldsymbol{s}}_0$ to the global coordinate frame of the celestial sky.
- Local coordinates are related to global coordinates by $\hat{\boldsymbol{s}}^{\mathrm{l}} = \mathbf{R}_0^{-1} \hat{\boldsymbol{s}}^{\mathrm{n}}$, where $\mathbf{R}_0$ is the 3×3 rotation matrix corresponding to the rotation $\mathcal{R}_0$.

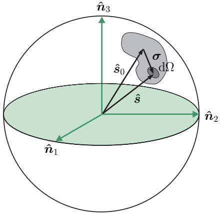

Figure: Geometry of observation of extended source.

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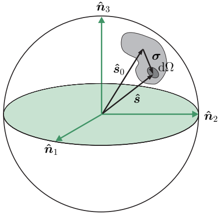

Figure: Geometry of observation of extended source.

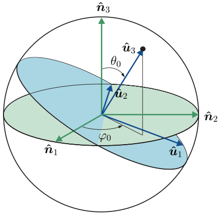

Figure: Rotation $\mathcal{R}_0$ mapping global coordinates of the celestial sky to local coordinates defined relative to the pointing direction $\hat{\boldsymbol{s}}_0$.

Coordinate systems

- Returning to the visibility function, we may now represent each function in its most natural coordinate system:
  - The beam function is most naturally represented in local coordinates relative to the pointing direction $\hat{\boldsymbol{s}}_0^{\mathrm{n}}$ and is denoted by $A^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}})$.
  - The source intensity function is most naturally represented in global coordinates and is denoted by $I^{\mathrm{n}}(\hat{\boldsymbol{s}}^{\mathrm{n}})$.

- We may convert function F^{n} in global coordinates to a corresponding function F^{l} in local coordinates through the rotation $\mathcal{R}_0$:

$$F^{\mathrm{n}}(\hat{\boldsymbol{s}}^{\mathrm{n}}) = F^{\mathrm{n}}(\mathbf{R}_0\hat{\boldsymbol{s}}^{\mathrm{l}}) = (\mathcal{R}_0^{-1}F^{\mathrm{n}})(\hat{\boldsymbol{s}}^{\mathrm{l}}) = F^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}}) , \quad \textit{i.e. } F^{\mathrm{l}} = \mathcal{R}_0^{-1}F^{\mathrm{n}} .$$

- The visibility integral may then be written

$$\mathcal{V}(\boldsymbol{u}) = \int_{\mathrm{S}^2} A^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}}) I^{\mathrm{n}}(\hat{\boldsymbol{s}}^{\mathrm{n}}) \mathrm{e}^{-\mathrm{i}2\pi\boldsymbol{u}\cdot\hat{\boldsymbol{s}}^{\mathrm{l}}} \,\mathrm{d}\Omega(\hat{\boldsymbol{s}}^{\mathrm{l}}) ,$$

  or in local coordinates

$$\mathcal{V}(\boldsymbol{u}) = \int_{\mathrm{S}^2} A^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}}) (\mathcal{R}_0^{-1}I^{\mathrm{n}})(\hat{\boldsymbol{s}}^{\mathrm{l}}) \mathrm{e}^{-\mathrm{i}2\pi\boldsymbol{u}\cdot\hat{\boldsymbol{s}}^{\mathrm{l}}} \,\mathrm{d}\Omega(\hat{\boldsymbol{s}}^{\mathrm{l}})$$
$$= \int_{\mathrm{S}^2} A^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}}) I^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}}) \mathrm{e}^{-\mathrm{i}2\pi\boldsymbol{u}\cdot\hat{\boldsymbol{s}}^{\mathrm{l}}} \,\mathrm{d}\Omega(\hat{\boldsymbol{s}}^{\mathrm{l}}) .$$

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Computing visibilities

- More general and accurate to compute full-sky visibilities in harmonic space.
- Substituting the harmonic expansion of the beam-modulated source intensity function $(A^{\mathrm{l}} \cdot I^{\mathrm{l}})(\hat{\boldsymbol{s}}^{\mathrm{l}}) = A^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}}) I^{\mathrm{l}}(\hat{\boldsymbol{s}}^{\mathrm{l}})$, visibility integral becomes

$$\mathcal{V}(\boldsymbol{u}) = \sum_{\ell m} \left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)_{\ell m} \int_{\mathrm{S}^2} \mathrm{e}^{-\mathrm{i}2\pi \boldsymbol{u} \cdot \hat{\boldsymbol{s}}^{\mathrm{l}}} Y_{\ell m}(\hat{\boldsymbol{s}}^{\mathrm{l}}) \, \mathrm{d}\Omega(\hat{\boldsymbol{s}}^{\mathrm{l}}) \, .$$

- Using the addition theorem for spherical harmonics, the Jacobi-Anger expansion and the orthogonality of the spherical harmonics the above integral can be evaluated analytically:

$$\int_{\mathrm{S}^2} \mathrm{e}^{-\mathrm{i}2\pi \boldsymbol{u} \cdot \hat{\boldsymbol{s}}^{\mathrm{l}}} Y_{\ell m}(\hat{\boldsymbol{s}}^{\mathrm{l}}) \, \mathrm{d}\Omega(\hat{\boldsymbol{s}}^{\mathrm{l}}) = 4\pi(-\mathrm{i})^{\ell} j_{\ell}(2\pi \|\boldsymbol{u}\|) Y_{\ell m}(\hat{\boldsymbol{u}}) \, .$$

- The harmonic representation of the full-sky visibility function then reads:

Harmonic representation of visibility

$$\mathcal{V}(\boldsymbol{u}) = 4\pi \sum_{\ell m} (-\mathrm{i})^{\ell} j_{\ell}(2\pi \|\boldsymbol{u}\|) Y_{\ell m}(\hat{\boldsymbol{u}}) \left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)_{\ell m}$$

Image reconstruction

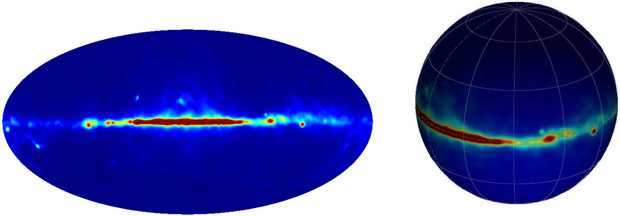
- Full-sky image reconstruction is possible in theory:

$$\int_{\mathrm{S}^2} \mathcal{V}(\boldsymbol{u}) Y^*_{\ell m}(\hat{\boldsymbol{u}})\, \mathrm{d}\Omega(\hat{\boldsymbol{u}}) = 4\pi(-\mathrm{i})^\ell j_\ell(2\pi\|\boldsymbol{u}\|)\left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)_{\ell m} .$$

- But not in practise since would require full sampling of the visibility function in $\mathbb{R}^3$.
- Instead use:
  - Standard Fourier transform approach for small patches.
  - w-projection (Cornwell *et al.* [2]) or faceting (*e.g.* Greisen [3]) approaches for wide fields of view.
- We consider only the forward problem of simulating visibilities in the full-sky setting and do not consider the reverse problem of image reconstruction any further.

Preliminary simulations

- Challenging computational problem.
- For a 50×50 pixel image of one square degree, require a harmonic band limit of $\ell_{\max} \simeq 13,000$.
- Solutions:
  - parallelise code;
  - fast methods such as wavelets (more to come on this).
- Present preliminary simulations here of mock observations of synchrotron emission (use synchrotron foreground map recovered from WMAP observations)



(a) Mollweide projection (b) Globe

Figure: Full-sky synchrotron map observed by WMAP and smoothed with a Gaussian kernel of $\mathrm{FWHM_S} = 1.7^{\circ}$.

Preliminary simulations

- Low resolution simulations: baseline limit of $u_{\max} = 30$; $\ell_{\max} \simeq 270$; reconstruct 20×20 image (corresponds to $\sim 20^\circ$ square patch).
- Rotate to local coordinates then compute visibilities for complete uv coverage, including full-sky contributions.
- Computations take $\sim$5 minutes on laptop (2.2GHz processor; 2GB RAM)
- Reconstructed image and tangent plane image match reasonably closely. Expected to differ slightly since:
  - full-sky contributions included when simulating visibilities but use Fourier transform to reconstruct image;
  - lose some high-frequency content due to low harmonic band-limit.

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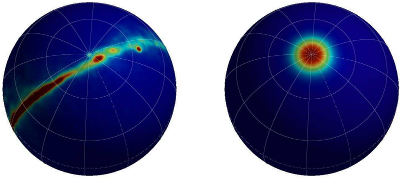

(a) Synchrotron map (b) Gaussian beam

Figure: Full-sky synchrotron and beam maps in local coordinates.

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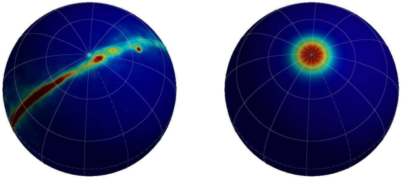

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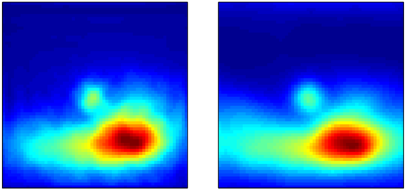

(a) Tangent plane (b) Full-sky simulation

Figure: Beam-modulated intensity images for a $\sim 20^\circ$ square patch.

Why wavelets?

Fourier (1807)

Haar (1909)

Morlet and Grossman (1981)

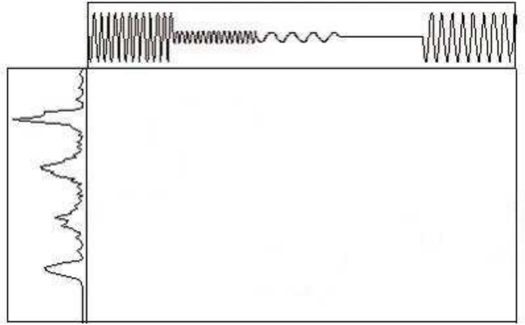

Figure: Fourier vs wavelet transform (image from `http://www.wavelet.org/tutorial/`)

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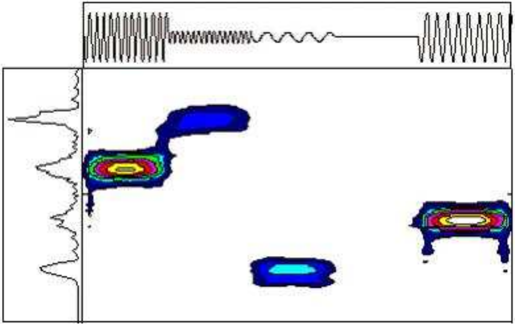

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Wavelets on the sphere

- Continuous wavelets on the sphere
  Antoine & Vandergheynst [1], Wiaux *et al.* [9], McEwen *et al.* [5]
  - Analysis:

$$\mathcal{W}_{\Psi}^{f}(a,\rho) = \int_{S^2} \mathrm{d}\Omega(\hat{\mathbf{s}})\, f(\hat{\mathbf{s}})\, \Psi_{a,\rho}^{*}(\hat{\mathbf{s}}) \ .$$

  - Synthesis:

$$f(\hat{\mathbf{s}}) = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) \int_{0}^{\infty} \frac{\mathrm{d}a}{a^3}\ \mathcal{W}_{\Psi}^{f}(a,\rho)\ [\mathcal{R}(\rho)\widehat{L}_{\Phi}\Psi_a](\hat{\mathbf{s}}) \ ,$$

  where $\mathrm{d}\varrho(\rho) = \sin\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma$ is the invariant measure on the rotation group $\mathrm{SO}(3)$ and $\widehat{L}_{\Phi}$ is a linear operator acting on the spherical harmonic coefficients of a function.

- Discrete wavelets on the sphere (multiresolution analysis)
  Schroder & Sweldens [7], McEwen & Eyers [4], Starck *et al.* [8], Wiaux, McEwen *et al.* [10]

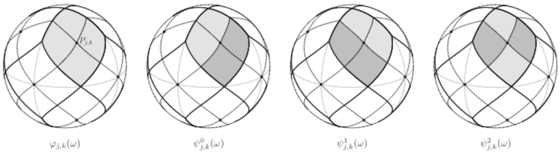

Figure: Haar scaling function and wavelets.

Wavelets on the sphere

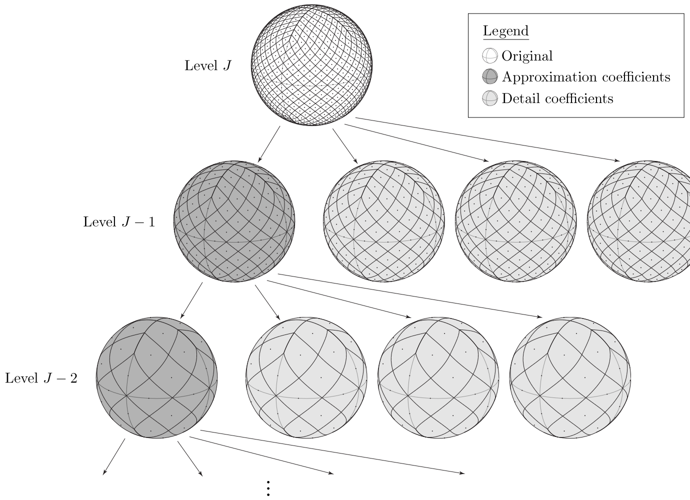

Figure: Haar multiresolution decomposition.

Fast wavelet methods

- Representing the beam-modulated intensity and the plane wave in an orthogonal wavelet basis on the sphere, with wavelets $\Psi_j(\hat{s}) \in \mathrm{L}^2(\mathrm{S}^2, \mathrm{d}\Omega)$:

$$\left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)(\hat{\boldsymbol{s}}^{\mathrm{l}}) = \sum_j \left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)_j \Psi_j(\hat{\boldsymbol{s}}^{\mathrm{l}}) \; ;$$

$$\mathrm{e}^{\mathrm{i}2\pi \boldsymbol{u}\cdot\hat{\boldsymbol{s}}^{\mathrm{l}}} = \sum_k E_k(\boldsymbol{u}) \Psi_k(\hat{\boldsymbol{s}}^{\mathrm{l}}) \; .$$

- Wavelet coefficients are given by the projection onto the wavelet basis functions:

$$\left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)_j = \int_{\mathrm{S}^2} \left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)(\hat{\boldsymbol{s}}^{\mathrm{l}}) \Psi_j^*(\hat{\boldsymbol{s}}^{\mathrm{l}}) \, \mathrm{d}\Omega(\hat{\boldsymbol{s}}^{\mathrm{l}}) \; ;$$

$$E_k(\boldsymbol{u}) = \int_{\mathrm{S}^2} \mathrm{e}^{\mathrm{i}2\pi \boldsymbol{u}\cdot\hat{\boldsymbol{s}}^{\mathrm{l}}} \, \Psi_k^*(\hat{\boldsymbol{s}}^{\mathrm{l}}) \, \mathrm{d}\Omega(\hat{\boldsymbol{s}}^{\mathrm{l}}) \; .$$

- Substituting these expansions into the visibility integral we find

$$\mathcal{V}(\boldsymbol{u}) = \sum_j \left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)_j E_j^*(\boldsymbol{u})$$

  where we have noted the orthogonality of the wavelet basis.

- Naive complexity of computing visibility for given $\boldsymbol{u}$ and $\hat{s}_0$, is $\mathcal{O}(J)$, where J is the number of basis functions ($\mathcal{O}(J) \sim \mathcal{O}({\ell_{\max}}^2)$ for the spherical harmonic basis).
- However, effective complexity reduced substantially by using a wavelet basis for which $\left(A^{\mathrm{l}} \cdot I^{\mathrm{l}}\right)$ is sparse.

Summary & future work

- Derived harmonic representation of visibility integral, including full-sky contributions.
- Framework allows extensions to complicated beams that depend on pointing position (although not discussed in this talk).
- Preformed very preliminary simulations to demonstrate and validate methodology.
- Future directions:
  - more realistic high-resolution simulations (parallelise implementation, incorporate extensions, incomplete uv coverage, evaluate effect of wide beam sidelobes);
  - fast wavelet methods to reduce computational requirements;
  - fast wavelet methods for wide field of view image reconstruction?

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