Radio interferometric imaging with compressive sensing

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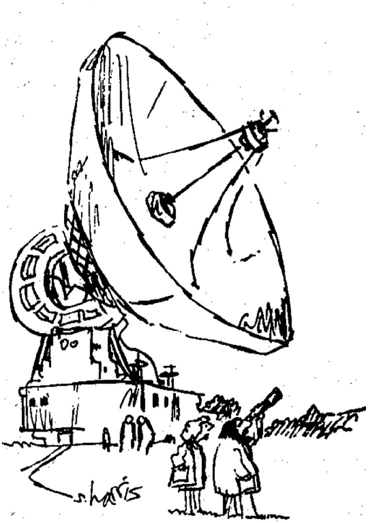
Radio telescopes are big!



"Just checking."

Radio telescopes are big!

Radio interferometric telescopes

Next-generation of radio interferometry rapidly approaching

- Square Kilometre Array (SKA) first observations planned for 2019.
- Many other pathfinder telescopes under construction, *e.g.* LOFAR, ASKAP, MeerKAT, MWA.
- Broad range of science goals.
- New modelling and imaging techniques required to ensure the next-generation of interferometric telescopes reach their full potential.

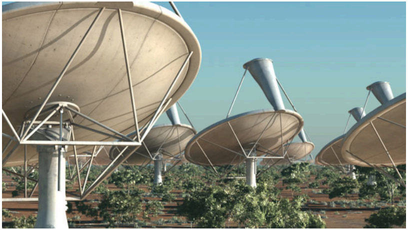


Figure: Artist impression of SKA dishes. [Credit: SKA Organisation]

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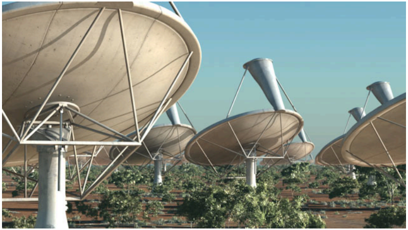

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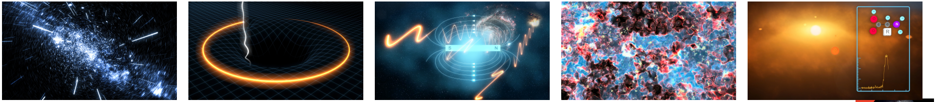

(a) Dark-energy (b) GR (c) Cosmic magnetism (d) Epoch of reionization (e) Exoplanets

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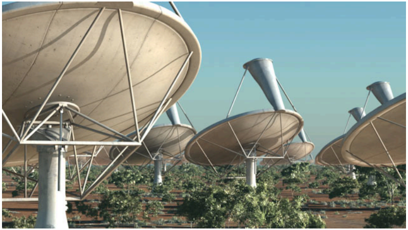

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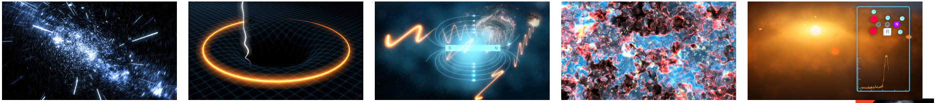

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Outline

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1. Inverse Problem
2. Interferometric Imaging with Compressive Sensing
3. Spread Spectrum Effect
4. Continuous Visibilities

Radio interferometric inverse problem

- The complex visibility measured by an interferometer is given by

$$\boxed{y(\boldsymbol{u}, w) = \int_{D^2} A(\boldsymbol{l})\, x(\boldsymbol{l})\, C(\|\boldsymbol{l}\|_2)\, \mathrm{e}^{-\mathrm{i}2\pi \boldsymbol{u}\cdot\boldsymbol{l}}\, \frac{\mathrm{d}^2\boldsymbol{l}}{n(\boldsymbol{l})}}\,,$$

visibilities

where the w-modulation $C(\|\boldsymbol{l}\|_2)$ is given by

$$\boxed{C(\|\boldsymbol{l}\|_2) \equiv \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}}\,.$$

w-modulation

- Various assumptions are often made regarding the size of the field-of-view:
  - Small-field with $\boxed{\|\boldsymbol{l}\|^2 w \ll 1}$ $\Rightarrow$ $\boxed{C(\|\boldsymbol{l}\|_2) \simeq 1}$
  - Small-field with $\boxed{\|\boldsymbol{l}\|^4 w \ll 1}$ $\Rightarrow$ $\boxed{C(\|\boldsymbol{l}\|_2) \simeq \mathrm{e}^{\mathrm{i}\pi w\|\boldsymbol{l}\|^2}}$
  - Wide-field $\Rightarrow$ $\boxed{C(\|\boldsymbol{l}\|_2) = \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}}$

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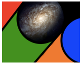

Radio interferometric inverse problem

- Consider the ill-posed inverse problem of radio interferometric imaging:

  $$\boxed{\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n}} \; ,$$

  where $\boldsymbol{y}$ are the measured visibilities, Φ is the linear measurement operator, $\boldsymbol{x}$ is the underlying image and $\boldsymbol{n}$ is instrumental noise.

- Measurement operator $\boxed{\Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A}}$ may incorporate:
  - primary beam $\mathbf{A}$ of the telescope;
  - w-modulation modulation $\mathbf{C}$;
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Interferometric imaging: recover an image from noisy and incomplete Fourier measurements.

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Interferometric imaging with compressed sensing

- Solve the interferometric imaging problem

$$\boxed{\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n} \quad \text{with} \quad \Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A}} \,,$$

by applying a prior on sparsity of the signal in a sparsifying dictionary Ψ.

- Basis pursuit (BP) denoising problem

$$\boxed{\boldsymbol{\alpha}^\star = \underset{\boldsymbol{\alpha}}{\arg\min} \|\boldsymbol{\alpha}\|_1 \;\; \text{such that} \;\; \|\boldsymbol{y} - \Phi\Psi\boldsymbol{\alpha}\|_2 \leq \epsilon \,,} \;\text{BPDN}$$

where the image is synthesised by $\boldsymbol{x}^\star = \Psi\boldsymbol{\alpha}^\star$.

- Total Variation (TV) denoising problem

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SARA for radio interferometric imaging

Algorithm

- Sparsity averaging reweighted analysis (SARA) for RI imaging (Carrillo, McEwen & Wiaux 2012)

- Consider a dictionary composed of a concatenation of orthonormal bases, i.e.

$$\Psi = \frac{1}{\sqrt{q}} [\Psi_1, \Psi_2, \ldots, \Psi_q],$$

  thus $\Psi \in \mathbb{R}^{N \times D}$ with $D = qN$.

- We consider the following bases: Dirac (*i.e.* pixel basis); Haar wavelets (promotes gradient sparsity); Daubechies wavelet bases two to eight.
  $\Rightarrow$ concatenation of 9 bases

- Promote average sparsity by solving the reweighted ℓ_1 analysis problem:

$$\min_{\bar{x} \in \mathbb{R}^N} \| W \Psi^T \bar{x} \|_1 \quad \text{subject to} \quad \| y - \Phi \bar{x} \|_2 \leq \epsilon \quad \text{and} \quad \bar{x} \geq 0 \,,$$

  SARA

  where $W \in \mathbb{R}^{D \times D}$ is a diagonal matrix with positive weights.

- Solve a sequence of reweighted ℓ_1 problems using the solution of the previous problem as the inverse weights $\rightarrow$ approximate the ℓ_0 problem.

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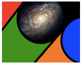
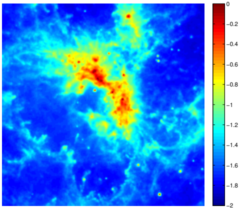


SARA for radio interferometric imaging

Results on simulations



(a) Original

SARA for radio interferometric imaging

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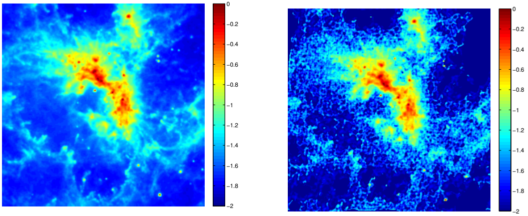

(a) Original (b) BP (SNR=16.67 dB)

SARA for radio interferometric imaging

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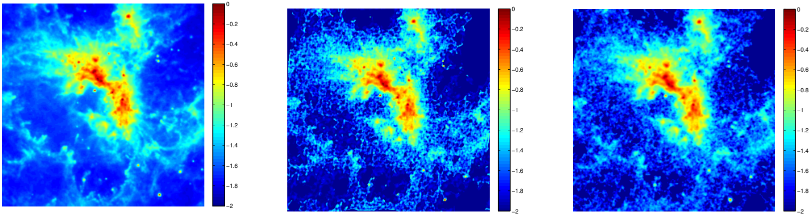

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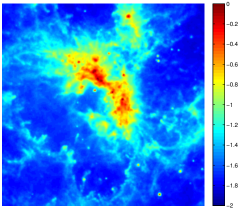

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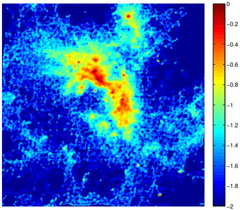

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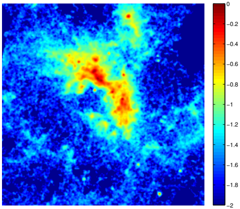

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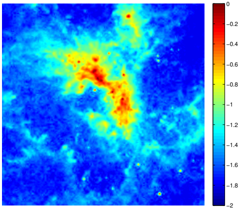
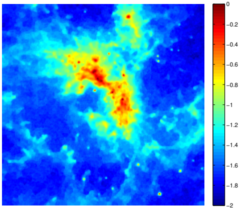
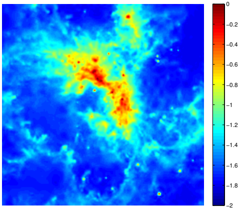


(d) BPDb8 (SNR=24.53 dB)



(e) TV (SNR=26.47 dB)



(f) SARA (SNR=29.08 dB)

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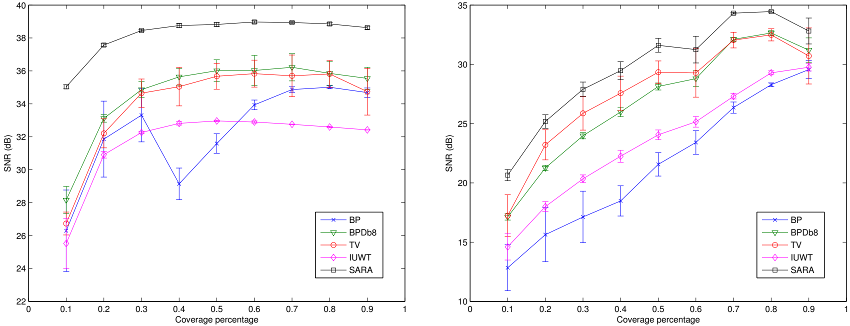

(a) M31

(b) 30Dor

Figure: Reconstruction fidelity vs visibility coverage.

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Spread spectrum effect

Preliminaries: sparsity and coherence

- What drives the quality of compressive sensing reconstruction?
- Number of measurements M required to achieve exact reconstruction given by

$$\boxed{M \geq c\mu^2 K \log N} \, ,$$

where K is the sparsity and N the dimensionality.

- Coherence between the measurement vectors and atoms of sparsity dictionary given by

$$\boxed{\mu = \sqrt{N} \max_{i,j} |\langle \Psi_i, \Phi_j \rangle|} \, .$$

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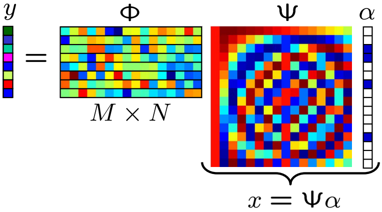

Spread spectrum effect

Review

- Non-coplanar baselines and wide fields → w-modulation → spread spectrum effect (first considered by Wiaux *et al.* 2009b).
- The w-modulation operator $\mathbf{C}$ has elements defined by

$$\boxed{C(l,m) \equiv \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-l^2-m^2}\right)} \simeq \mathrm{e}^{\mathrm{i}\pi w \|\boldsymbol{l}\|^2} \quad \text{for} \quad \|\boldsymbol{l}\|^4\, w \ll 1}\ ,$$

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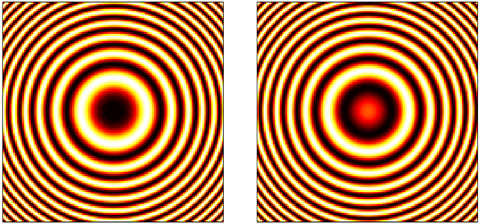

(a) Real part (b) Imaginary part

Figure: Chirp modulation.

Spread spectrum effect

Review

Spread spectrum effect in a nutshell

1. Radio interferometers take (essentially) Fourier measurements.
2. Recall, the coherence is the maximum inner product between measurement vectors and sparsifying atoms.
3. Thus, coherence is (essentially) the maximum of the Fourier coefficients of the atoms of the sparsifying dictionary.
4. w-modulation spreads the spectrum of the atoms of the sparsifying dictionary, reducing the maximum Fourier coefficient.
5. Spreading the spectrum reduces coherence, thus improving reconstruction fidelity.

- Consistent with findings of Carozzi et al. (2013) from information theoretic approach.
- Studied for constant w (for simplicity) by Wiaux *et al.* (2009b).
- Studied for varying w (with realistic images and various sparse representations) by Wolz *et al.* (2013).

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Spread spectrum effect in a nutshell

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Spread spectrum effect

Sparse w-projection

- Apply the w-projection algorithm (Cornwell *et al.* 2008) to shift the w-modulation through the Fourier transform:

$$\boxed{\Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A} \quad \Rightarrow \quad \Phi = \hat{\mathbf{C}}\,\mathbf{F}\,\mathbf{A}} \, .$$

- Naively, expressing the application of the w-modulation in this manner is computationally less efficient that the original formulation but it has two important advantages.
- Different w for each (u, v), while still exploiting FFT.
- Many of the elements of $\hat{\mathbf{C}}$ will be close to zero.
- Support of w-modulation in Fourier space determined dynamically.

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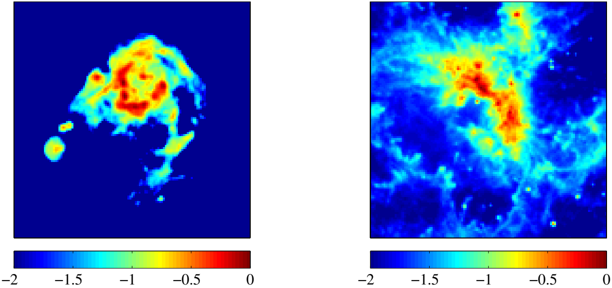
Spread spectrum effect for varying w

Results on simulations

- Perform simulations to assess the effectiveness of the spread spectrum effect in the presence of varying w.
- Consider idealised simulations with uniformly random visibility sampling.

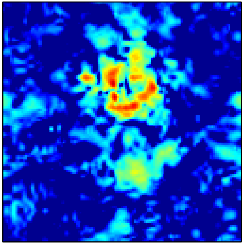


(a) M31 (b) 30Dor

Figure: Ground truth images in logarithmic scale.

Spread spectrum effect for varying w

Results on simulations



(a) $w_{\mathrm{d}} = 0 \rightarrow$ SNR$=5$dB

Figure: Reconstructed images of M31 for 10% coverage.

Spread spectrum effect for varying w

Results on simulations

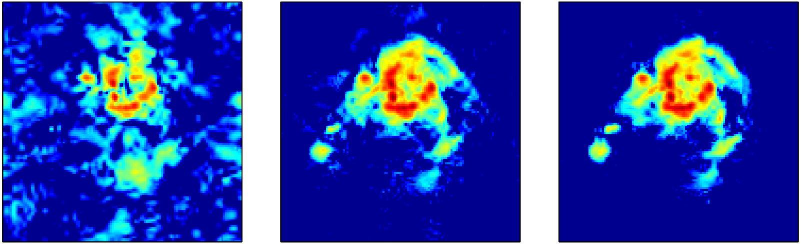
(a) $w_{\mathrm{d}} = 0 \rightarrow$ SNR$=$ 5dB

(c) $w_{\mathrm{d}} = 1 \rightarrow$ SNR$=$ 19dB

Figure: Reconstructed images of M31 for 10% coverage.

Spread spectrum effect for varying w

Results on simulations



(a) $w_\mathrm{d} = 0 \rightarrow$ SNR$=$ 5dB (b) $w_\mathrm{d} \sim \mathcal{U}(0, 1) \rightarrow$ SNR$=$ 16dB (c) $w_\mathrm{d} = 1 \rightarrow$ SNR$=$ 19dB

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Spread spectrum effect for varying w

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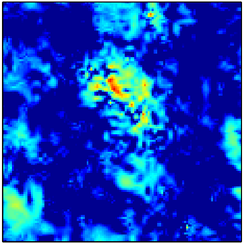

(a) $w_{\rm d} = 0 \rightarrow$ SNR$=2$dB

Figure: Reconstructed images of 30Dor for 10% coverage.

Spread spectrum effect for varying w

Results on simulations

(a) $w_{\rm d} = 0 \rightarrow$ SNR$=$ 2dB

(c) $w_{\rm d} = 1 \rightarrow$ SNR$=$ 15dB

Figure: Reconstructed images of 30Dor for 10% coverage.

Spread spectrum effect for varying w

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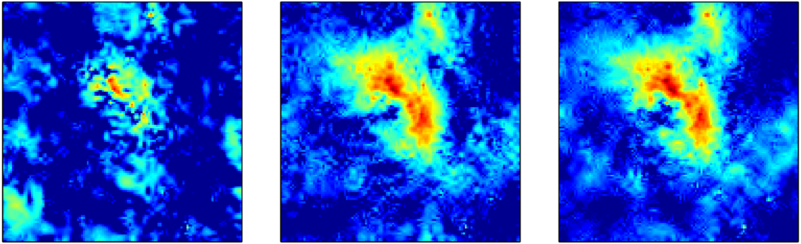

(a) $w_\mathrm{d} = 0 \rightarrow$ SNR$=$ 2dB (b) $w_\mathrm{d} \sim \mathcal{U}(0, 1) \rightarrow$ SNR$=$ 12dB (c) $w_\mathrm{d} = 1 \rightarrow$ SNR$=$ 15dB

Figure: Reconstructed images of 30Dor for 10% coverage.

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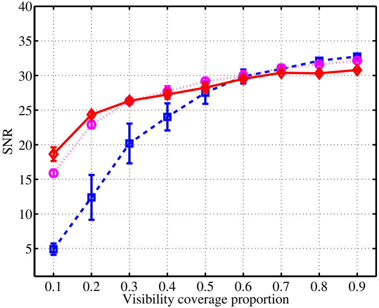
Results on simulations



(a) Daubechies 8 (Db8) wavelets

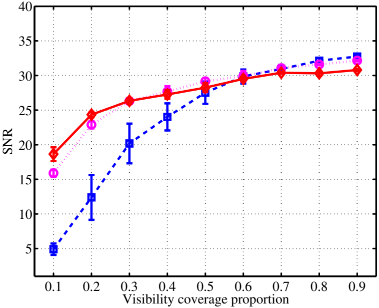
Figure: Reconstruction fidelity for M31.

Improvement in reconstruction fidelity due to the spread spectrum effect for varying w is almost as large as the case of constant maximum w!

- As expected, for the case where coherence is already optimal, there is little improvement.

Spread spectrum effect for varying w

Results on simulations



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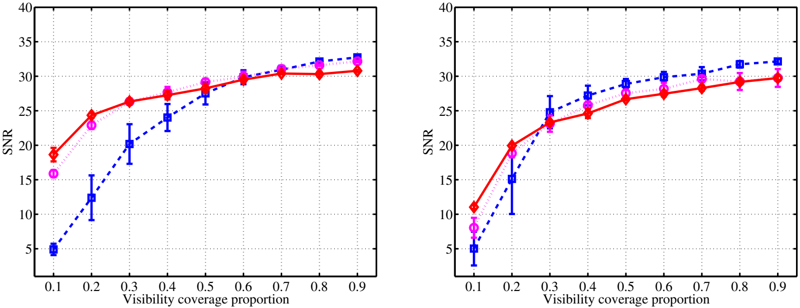

(a) Daubechies 8 (Db8) wavelets (b) Dirac basis

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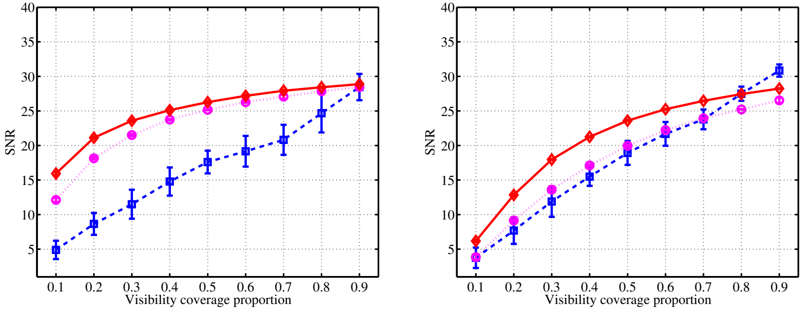

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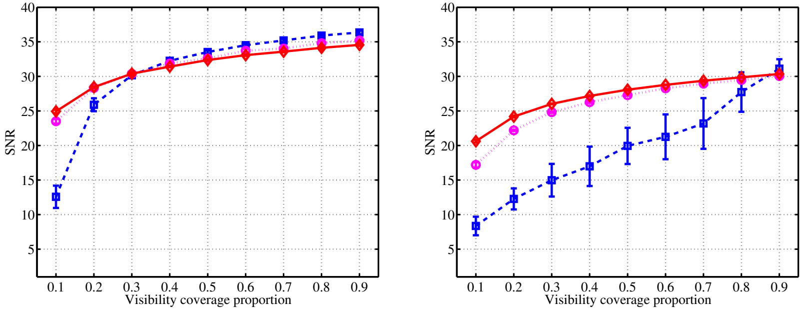

(a) M31 (b) 30 Dor

Figure: Reconstruction fidelity using SARA.

Improvement in reconstruction fidelity due to the spread spectrum effect for varying w is almost as large as the case of constant maximum w!

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Outline

1. Inverse Problem
2. Interferometric Imaging with Compressive Sensing
3. Spread Spectrum Effect
4. Continuous Visibilities

Supporting continuous visibilities

Algorithm

- Ideally we would like to model the continuous Fourier transform operator

$$\boxed{\Phi = \mathbf{F}^{\mathrm{c}}} \, .$$

- But this is impractically slow!
- Incorporated gridding into our CS interferometric imaging framework (Carrillo *et al.* 2013).
- Model with measurement operator

$$\boxed{\Phi = \mathbf{G}\,\mathbf{F}\,\mathbf{D}\,\mathbf{Z}} \, ,$$

where we incorporate:

  - convolutional gridding operator $\mathbf{G}$;
  - fast Fourier transform $\mathbf{F}$;
  - normalisation operator $\mathbf{D}$ to undo the convolution gridding;
  - zero-padding operator $\mathbf{Z}$ to upsample the discrete visibility space.

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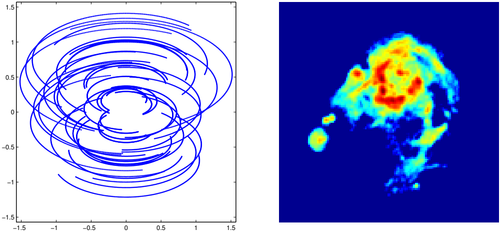

(a) Coverage

(b) M31 (ground truth)

Figure: Reconstructed images from continuous visibilities.

Supporting continuous visibilities

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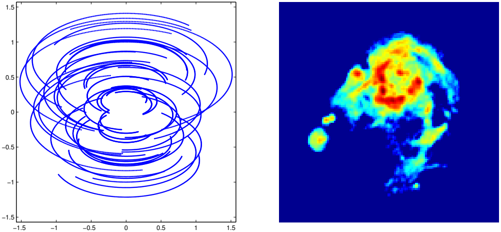

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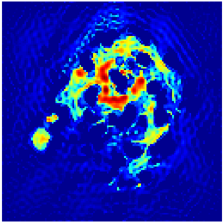

(c) Dirac basis → SNR= 8.2dB

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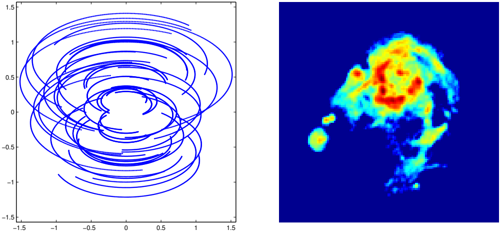

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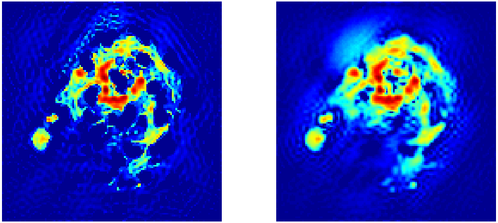

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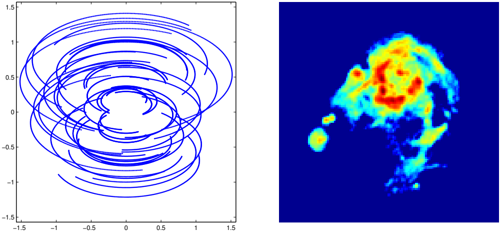

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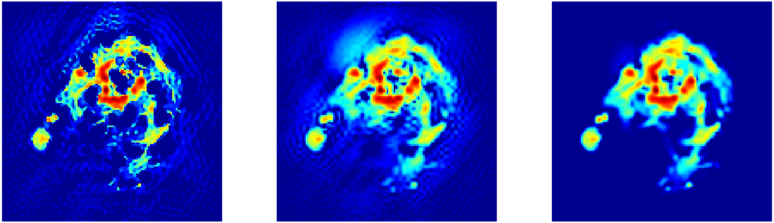

(c) Dirac basis → SNR= 8.2dB (d) Db8 wavelets → SNR= 11.1dB (e) SARA → SNR= 13.4dB

Figure: Reconstructed images from continuous visibilities.

Conclusions & outlook

- Effectiveness of compressive sensing for radio interferometric imaging demonstrated.
- Theory of compressive sensing can guide telescope design.
- We have just released the PURIFY code to scale to the realistic setting.
- Includes state-of-the-art convex optimisation algorithms that support parallelisation.
- Application of (unconstrained) BPDN to LOFAR by Garsden *et al.* 2014.

Apply to observations made by real interferometric telescopes.

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Extra Slides

Sparse w-projection

Sparsity of w-modulation kernel in Fourier space

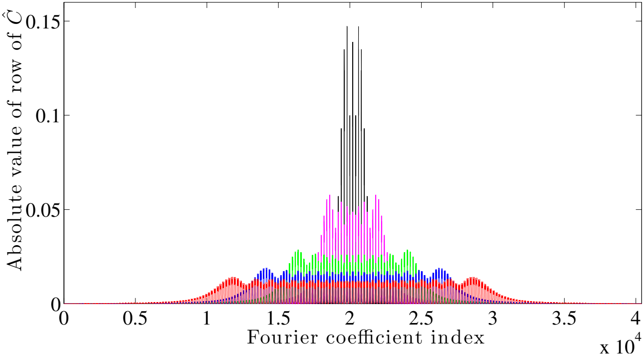

Figure: Rows of Fourier representation of w-modulation operator $\hat{\mathbf{C}}$.

Sparse w-projection

Dynamic sparsification

- We make a sparse matrix approximation of $\hat{\mathsf{C}}$ to speed up its computational application and reduce memory requirements.
- Sparsify $\hat{\mathsf{C}}$ by dynamic thresholding.
- Retain $E\%$ of the energy content for each visibility measurement.
- Support of w-modulation kernel in Fourier space determined dynamically, so don't require any prior information about structure.
- Generic procedure applicable for any direction-dependent effect (DDE).

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Sparsified w-modulation kernels

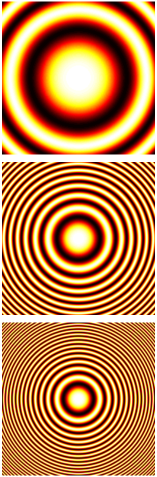
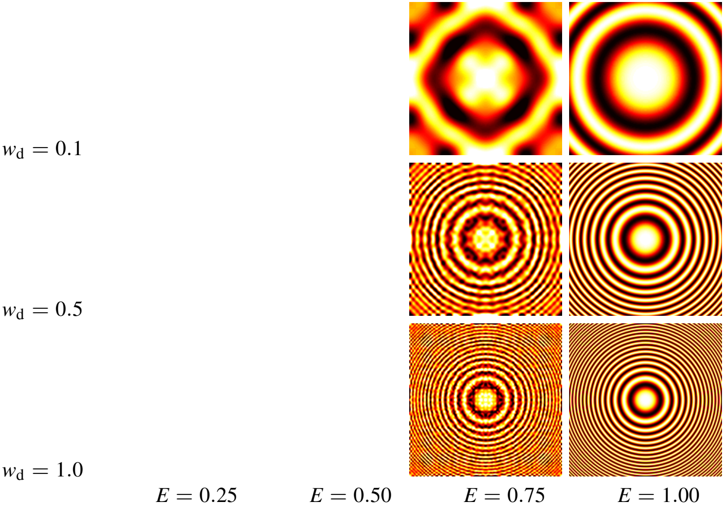


Figure: w-modulation kernel.

Sparse w-projection

Sparsified w-modulation kernels



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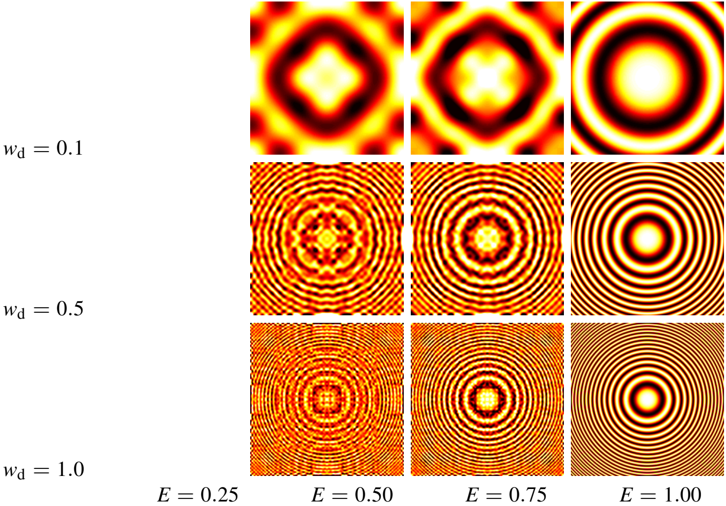

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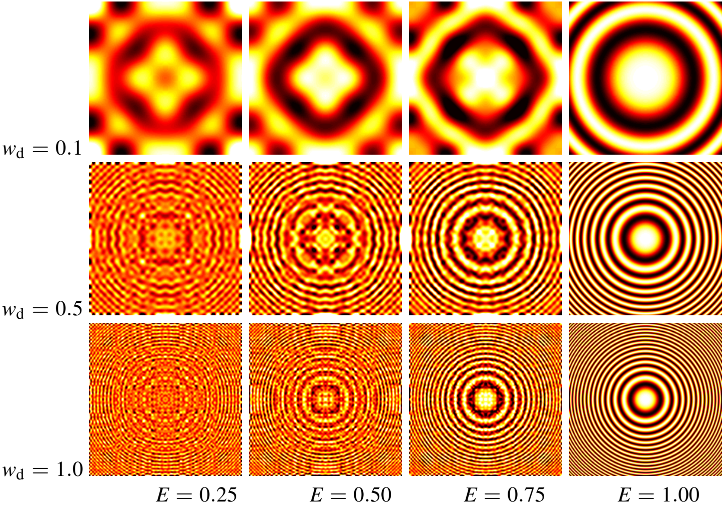

Figure: w-modulation kernel.

Sparse w-projection

Proportion of non-zero entries

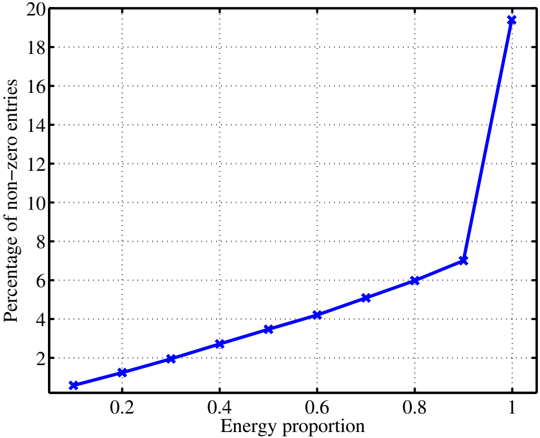

Figure: Percentage of non-zero entries as a function of preserved energy proportion.

Sparse w-projection

Runtime improvements

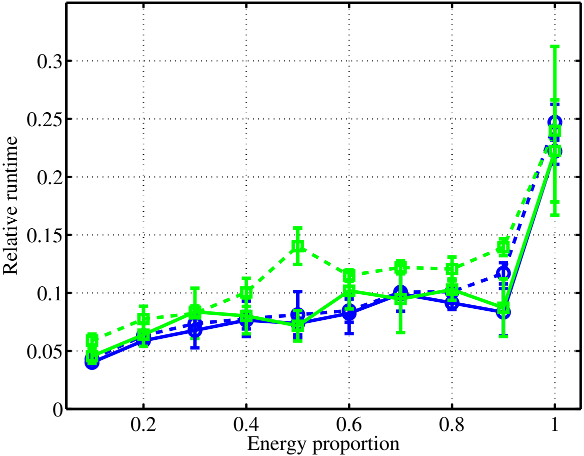

Figure: Relative runtime as a function of preserved energy proportion for 10% (dashed) and 50% (solid) visibility coverages.

Sparse w-projection

Impact on reconstruction quality

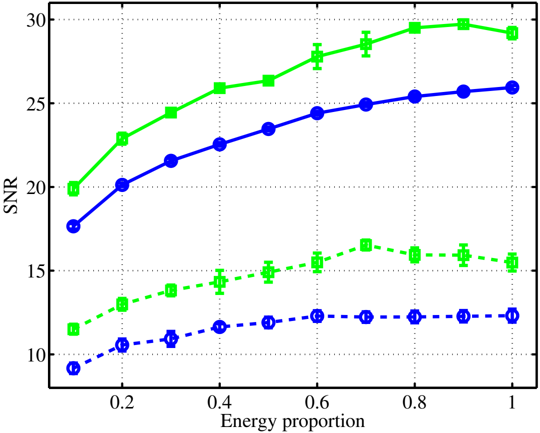

Figure: Reconstruction quality as a function of preserved energy proportion for 10% (dashed) and 50% (solid) visibility coverages.