# Statistical approaches for sparse radio interferometric imaging

## Error estimation in radio interferometry imaging
## or
## "Bayesian compressive sensing"

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CALIM 2016, Socorro, October 2016

# Outline

1. Bayesian interpretations of interferometric imaging techniques
2. Proximal MCMC sampling
3. Sparse regularisation with Bayesian credible regions

# Outline

# Radio interferometric inverse problem

- Consider the ill-posed inverse problem of radio interferometric imaging:

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{n},$$

where $\boldsymbol{y}$ are the measured visibilities, $\boldsymbol{\Phi}$ is the linear measurement operator, $\boldsymbol{x}$ is the underlying image and $\boldsymbol{n}$ is instrumental noise.

- Measurement operator, *e.g.* $\Phi = \mathsf{GFA}$, may incorporate:
  - primary beam $\mathsf{A}$ of the telescope;
  - Fourier transform $\mathsf{F}$;
  - convolutional de-gridding $\mathsf{G}$ to interpolate to continuous $uv$-coordinates;
  - direction-dependent effects (DDEs)...

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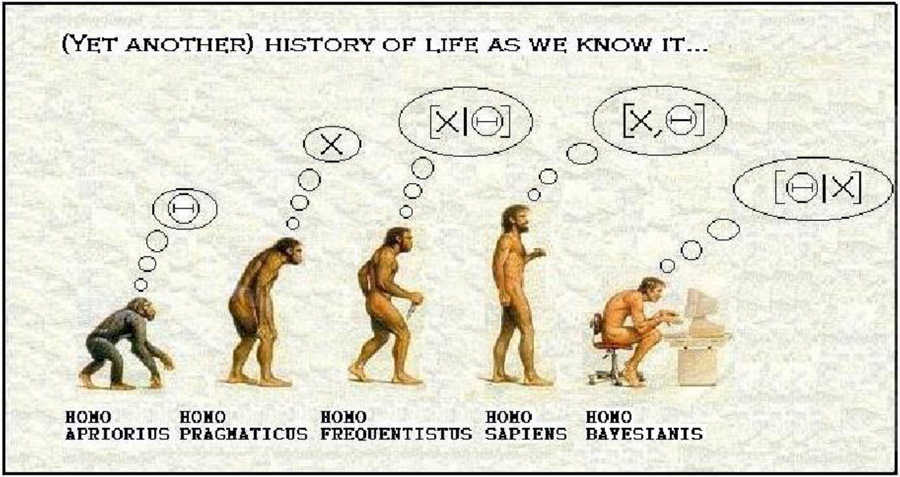
Interferometric imaging: recover an image from noisy and incomplete Fourier measurements.

# Bayesian evolution



# Bayesian inference

- Given data $\boldsymbol{y}$ (visibilities) and model $M$ (interferometric telescope with Gaussian noise), we want a full probabilistic description of our knowledge of the underlying sky image $\boldsymbol{x}$.

- Bayes to the rescue:

$$\mathrm{P}(\boldsymbol{x} \mid \boldsymbol{y}, M) = \frac{\mathrm{P}(\boldsymbol{y} \mid \boldsymbol{x}, M)\, \mathrm{P}(\boldsymbol{x} \mid M)}{\mathrm{P}(\boldsymbol{y} \mid M)}$$

Bayes Theorem

- Bayes theorem in words:

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

- How do we perform Bayesian inference in practice?

  ⇒ maximum a-posteriori (MAP) estimates and sampling approaches (MCMC)

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# Bayes in practice

## MAP and MCMC sampling

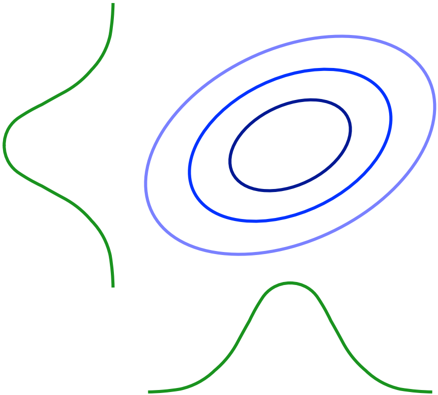

Figure: Probability distribution to explore in 2D

# Bayes in practice

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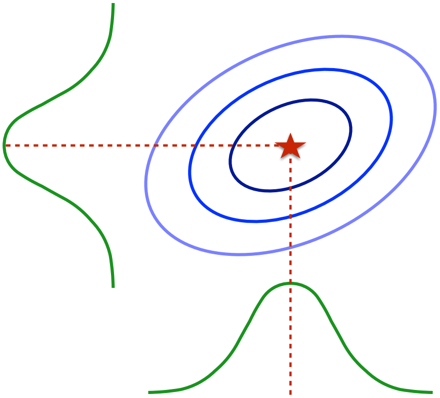

Figure: Maximum a-posteriori (MAP) estimate

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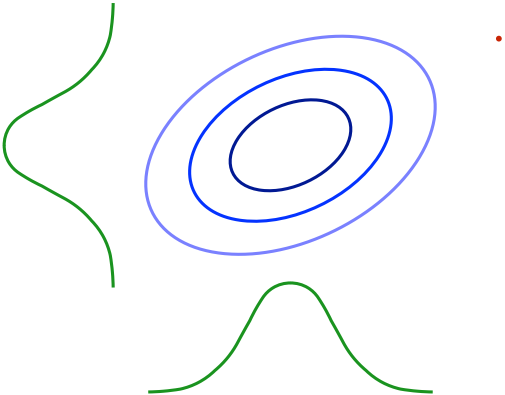

Figure: Markov Chain Monte Carlo (MCMC) sampling

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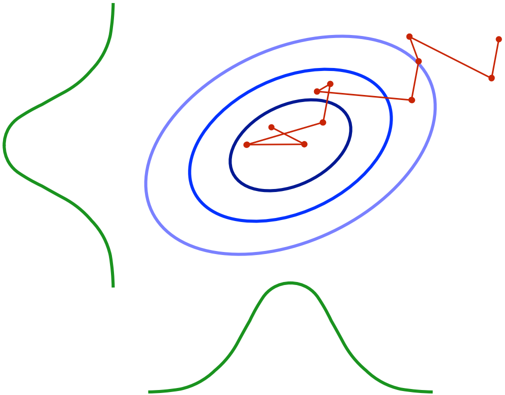

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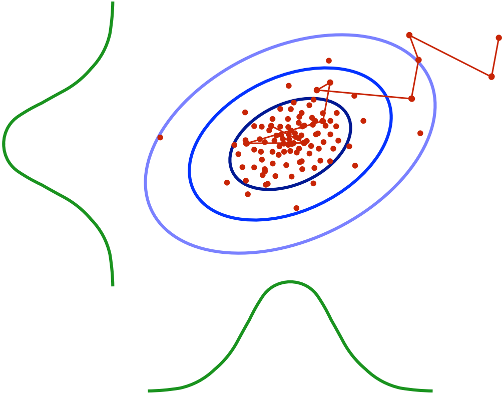

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# MAP estimation and regularisation

## Hint: they're the same thing!

- Many interferometric imaging approaches are based on regularisation (*i.e.* minimising an objective function comprised of a data-fidelity penalty and a regularisation penalty).
- Consider the MAP estimation problem. . .

# MAP estimation and regularisation

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- Start with Bayes Theorem (ignore normalising evidence):

$$\mathrm{P}(\boldsymbol{x} \mid \boldsymbol{y}) \propto \mathrm{P}(\boldsymbol{y} \mid \boldsymbol{x}) \mathrm{P}(\boldsymbol{x}) \,, \quad \textit{i.e.} \text{ posterior} \propto \text{likelihood} \times \text{prior}$$

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Define likelihood (assuming Gaussian noise) and prior:

$$\mathrm{P}(\boldsymbol{y} \mid \boldsymbol{x}) \propto \exp\left(-\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2/(2\sigma^2)\right)$$

Likelihood

- MAP estimator:

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$$\mathrm{P}(\boldsymbol{x}) \propto \exp\left(-R(\boldsymbol{x})\right)$$

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$$\log \mathrm{P}(\boldsymbol{x} \mid \boldsymbol{y}) = -\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2/(2\sigma^2) - R(\boldsymbol{x}) + \text{const.}$$

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# CLEAN and MEM as MAP estimators

- CLEAN

  Consider the sparse prior: $\mathrm{P}(\boldsymbol{x}) \propto \exp\left(-\beta \left\|\boldsymbol{x}\right\|_0\right)$.

  Corresponding MAP estimator is:

$$\boldsymbol{x}_{\text{clean}} = \arg\min_{\boldsymbol{x}} \left[ \left\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\right\|_2^2 + \lambda \left\|\boldsymbol{x}\right\|_0 \right]$$

  (Laplace prior $\mathrm{P}(\boldsymbol{x}) \propto \exp\left(-\beta \left\|\boldsymbol{x}\right\|_1\right)$ is good proxy; Wiaux *et al.* 2009)

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(In practice some differences: CLEAN does not solve MAP problem exactly; MEM considered in RI imposes additional constraints.)

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# Compressive sensing

## Synthesis framework

- Consider sparsifying representation (*e.g.* wavelet basis):

$$\boldsymbol{x} = \sum_i \Psi_i \alpha_i = \begin{pmatrix} | \\ \Psi_0 \\ | \end{pmatrix} \alpha_0 + \begin{pmatrix} | \\ \Psi_1 \\ | \end{pmatrix} \alpha_1 + \cdots \quad \Rightarrow \quad \boxed{\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\alpha}}$$

- Recover (wavelet) coefficients $\boldsymbol{\alpha}$ of image $\boldsymbol{x}$.
- Consider the Laplacian prior on coefficients: $\mathrm{P}(\boldsymbol{\alpha}) \propto \exp\left(-\beta \left\|\boldsymbol{\alpha}\right\|_1\right)$.
- Sparse synthesis regularisation problem:

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## More sophisticated approaches

- Overcomplete dictionary composed of a concatenation of orthonormal bases:

$$\boldsymbol{\Psi} = \left[\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \ldots, \boldsymbol{\Psi}_q\right]$$

- Constrained vs unconstrained problems.
- Re-weighted versions and the SARA algorithm (Carrillo, McEwen, Wiaux 2012).
- Sparse analysis regularisation problem:

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Analysis framework

- Analysis problem viewed from a synthesis perspective:

$$\boldsymbol{x}_{\text{analysis}} = \boldsymbol{\Psi} \times \arg\min_{\boldsymbol{\gamma} \in \text{column space}(\boldsymbol{\Psi}^\dagger)} \left[ \left\| \boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{\Psi}^\dagger \boldsymbol{\gamma} \right\|_2^2 + \lambda \left\| \boldsymbol{\gamma} \right\|_1 \right]$$

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# Sampling the full posterior distribution
## Markov Chain Monte Carlo (MCMC)

- Alternative is to sample full posterior distribution $\mathrm{P}(\boldsymbol{x} \mid \boldsymbol{y})$.
- Provides uncertainly (error) information.
- MCMC methods for high-dimensional problems (like interferometric imaging):
  - Gibbs sampling (sample from conditional distributions)
  - Hamiltonian MC (HMC) sampling (exploit gradients)
  - Metropolis adjusted Langevin algorithm (MALA) sampling (exploit gradients)
- Gibbs sampling applied to radio interferometric imaging (Sutter, Wandelt, McEwen, *et al.* 2014), using methods developed for CMB by Wandelt *et al.* (2005).
  - Assume isotropic Gaussian process prior characterised by power spectrum $C_\ell$.
  - Sample from conditional distributions:

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# Sampling the full posterior distribution
## Markov Chain Monte Carlo (MCMC)

- Alternative is to sample full posterior distribution $\mathrm{P}(\boldsymbol{x} \mid \boldsymbol{y})$.
- Provides uncertainly (error) information.
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    - Gibbs sampling (sample from conditional distributions)
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Require MCMC approach to support sparse priors, which shown to be highly effective.

# Outline

# MCMC sampling with gradients
## Langevin dynamics

- Work done by Xiaohao Cai in collaboration with Marcelo Pereyra.
- Consider posteriors of the following form (and more compact notation):

$$\mathrm{P}(\boldsymbol{x} \mid \boldsymbol{y}) = \underbrace{\pi(\boldsymbol{x})}_{\text{Posterior}} \propto \exp\big[-\underbrace{g(\boldsymbol{x})}_{\text{Convex}}\big]$$

- If $g(\boldsymbol{x})$ differentiable can adopt MALA (Langevin dynamics) or HMC (Hamiltonian dynamics) MCMC methods.
- Langevin dynamics model molecular dynamics (includes friction and occasional high velocity collisions that perturb the system).
- Based on Langevin diffusion process $\mathcal{L}(t)$, with $\pi$ as stationary distribution:

$$\mathrm{d}\mathcal{L}(t) = \frac{1}{2}\nabla \log \pi\big(\mathcal{L}(t)\big)\mathrm{d}t + \mathrm{d}\mathcal{W}(t), \quad \mathcal{L}(0) = l_0$$

where $\mathcal{W}$ is Brownian motion.
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- Need gradients so cannot support sparse priors.

# Proximity operators

## A brief aside

- Define proximity operator:

$$\mathrm{prox}_g^{\lambda}(\boldsymbol{x}) = \arg\min_{\boldsymbol{u}} \left[ g(\boldsymbol{u}) + \|\boldsymbol{u} - \boldsymbol{x}\|^2 / 2\lambda \right]$$

- Generalisation of projection operator:

$$\mathcal{P}_{\mathcal{C}}(\boldsymbol{x}) = \arg\min_{\boldsymbol{u}} \left[ \imath_{\mathcal{C}}(\boldsymbol{u}) + \|\boldsymbol{u} - \boldsymbol{x}\|^2 / 2 \right],$$

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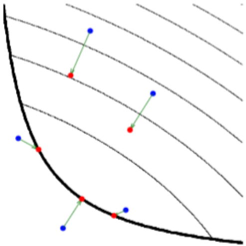

Figure: Illustration of proximity operator [Credit: Parikh & Boyd (2013)]

# Proximal MCMC

## Moreau approximation

- Follow Pereyra (2016a) and consider Moreau approximation of $\pi$:

$$\pi_\lambda(\boldsymbol{x}) = \sup_{\boldsymbol{u}\in\mathbb{R}^N} \pi(\boldsymbol{u}) \exp\Big(-\frac{\|\boldsymbol{u}-\boldsymbol{x}\|^2}{2\lambda}\Big)$$

- Important properties of $\pi_\lambda(\boldsymbol{x})$:

  1. As $\lambda \rightarrow 0$, $\pi_\lambda(\boldsymbol{x}) \rightarrow \pi(\boldsymbol{x})$
  2. $\nabla \log \pi_\lambda(\boldsymbol{x}) = (\text{prox}_g^\lambda(\boldsymbol{x}) - \boldsymbol{x})/\lambda \in \partial \log \pi\big(\text{prox}_g^\lambda(\boldsymbol{x})\big)$

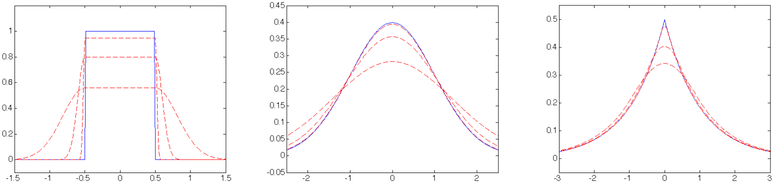

Figure: Illustration of Moreau approximations [Credit: Pereyra (2016a)]

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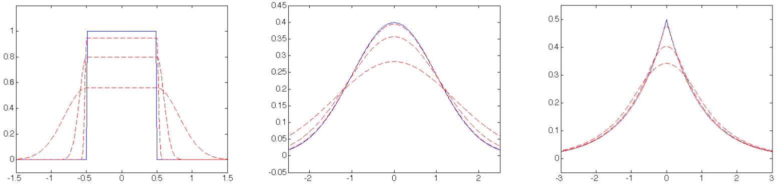


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# Proximal MCMC

## P-MALA

- Discretise Langevin differential equation using forward Euler approximation:

$$\boldsymbol{l}^{(m+1)} = \boldsymbol{l}^{(m)} + \frac{\delta}{2} \nabla \log \pi(\boldsymbol{l}^{(m)}) + \sqrt{\delta} \boldsymbol{w}^{(m)},$$

  where $\delta$ controls the discrete-time increment and $\boldsymbol{w}^{(m)}$ is (unit) Gaussian distributed.

- Apply Moreau approximation and compute gradient by prox:

$$\boldsymbol{l}^{(m+1)} = \operatorname{prox}_g^{\delta/2}(\boldsymbol{l}^{(m)}) + \sqrt{\delta} \boldsymbol{w}^{(m)}.$$

- (After adding a Metropolis-Hastings accept-reject step, get P-MALA with $\boldsymbol{l}^{(m)} \to \pi$.)

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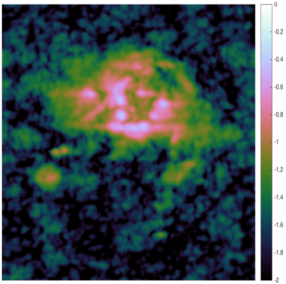


(a) Dirty image

Figure: HII region of M31

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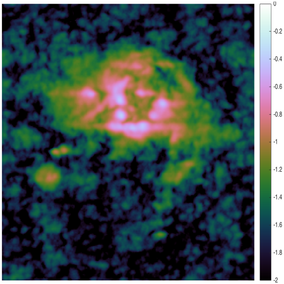


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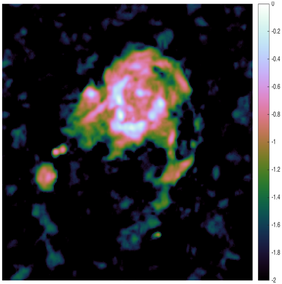


(b) Mean recovered image

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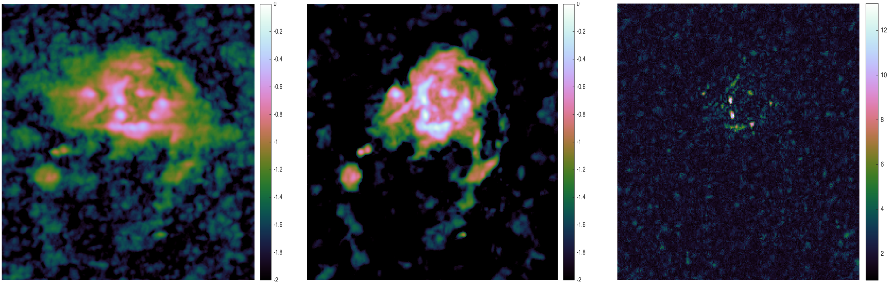


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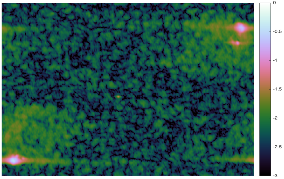


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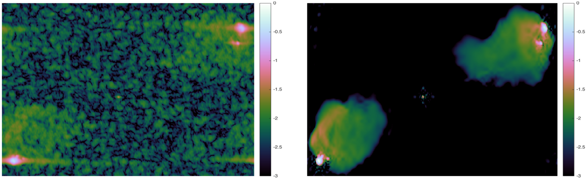


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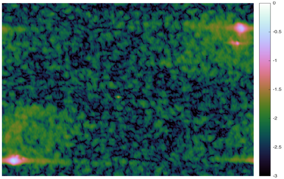


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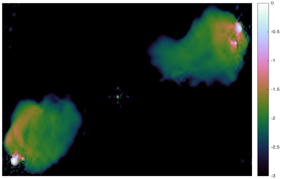


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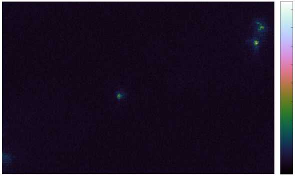

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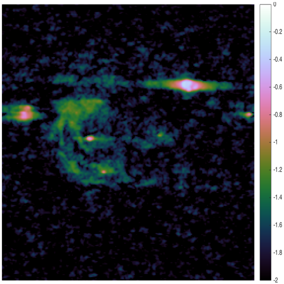


(a) Dirty image

Figure: Supernova remnant W28

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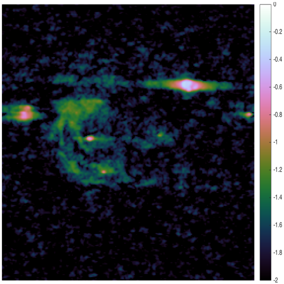


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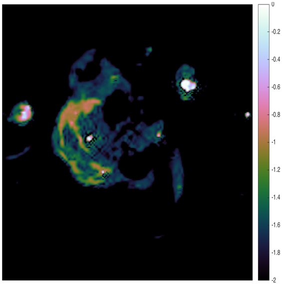


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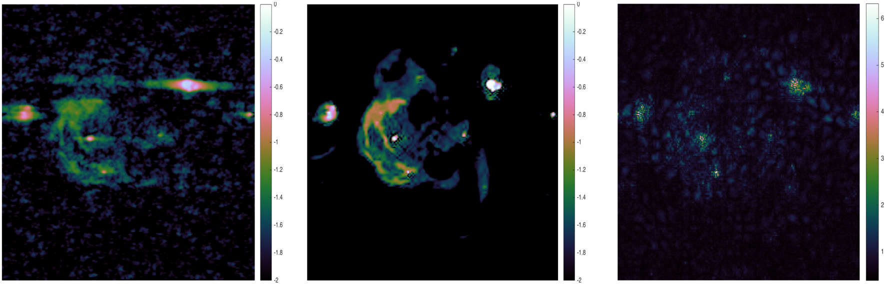


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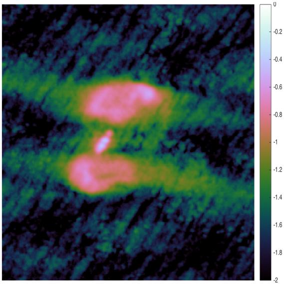


(a) Dirty image

Figure: 3C288

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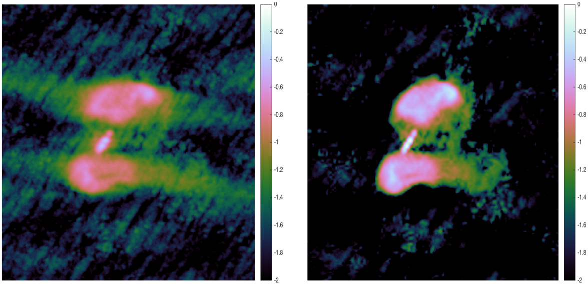


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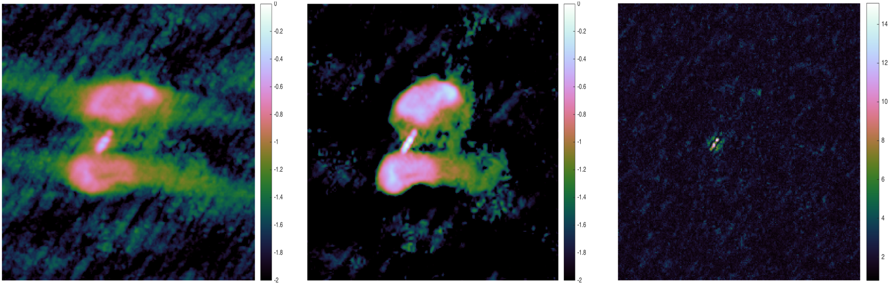


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# Outline

1. Bayesian interpretations of interferometric imaging techniques
2. Proximal MCMC sampling
3. Sparse regularisation with Bayesian credible regions

# Bayesian credible regions

## Computation

- Combine error estimation with fast sparse regularisation (*cf.* compressive sensing) methods.
- Let $C_\alpha$ denote a Bayesian credible region with confidence level $(1-\alpha)\%$:

$$\mathrm{P}(\boldsymbol{x} \in C_\alpha \,|\, \boldsymbol{y}) = \int_{\boldsymbol{x} \in C_\alpha} p(\boldsymbol{x}|\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x} = 1 - \alpha$$

- Define $C_\alpha$ by posterior iso-contour:

$$C_\alpha := \{\boldsymbol{x} : g(\boldsymbol{x}) \leq \gamma_\alpha\}$$

- Analytic approximation of $\gamma_\alpha$ derived in Pereyra (2016b):

$$\tilde{\gamma}_\alpha = g_{\boldsymbol{y}}(\boldsymbol{x}_{\mathrm{map}}) + N(\tau_\alpha + 1),$$

  where $\tau_\alpha = \sqrt{16 \log(3/\alpha)/N}$ and $\alpha \in (4\exp(-N/3), 1)$.

- Compute $\boldsymbol{x}_{\mathrm{map}}$ by sparse regularisation and estimate Bayesian credible regions.

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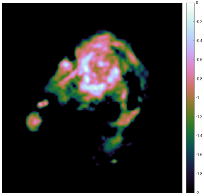
$$\tilde{\gamma}_\alpha = g_{\boldsymbol{y}}(\boldsymbol{x}_{\mathrm{map}}) + N(\tau_\alpha + 1) ,$$

  where $\tau_\alpha = \sqrt{16 \log(3/\alpha)/N}$ and $\alpha \in (4\exp(-N/3), 1)$.

- Compute $\boldsymbol{x}_{\mathrm{map}}$ by sparse regularisation and estimate Bayesian credible regions.

# Bayesian credible regions

## Preliminary results on simulations

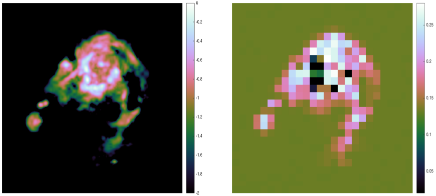



(a) Recovered image

Figure: HII region of M31

# Bayesian credible regions

## Preliminary results on simulations



(a) Recovered image

(b) Credible intervals for regions of size $10 \times 10$

Figure: HII region of M31

# Bayesian credible regions

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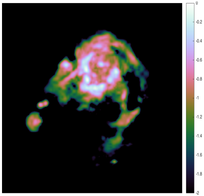

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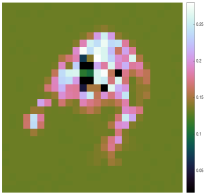

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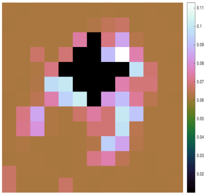
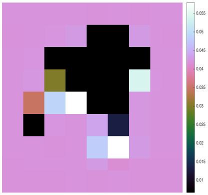


(c) Credible intervals for regions of size $20 \times 20$



(d) Credible intervals for regions of size $30 \times 30$

Figure: HII region of M31

# Bayesian credible regions

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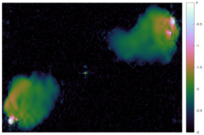

(a) Recovered image

Figure: Cygnus A

# Bayesian credible regions

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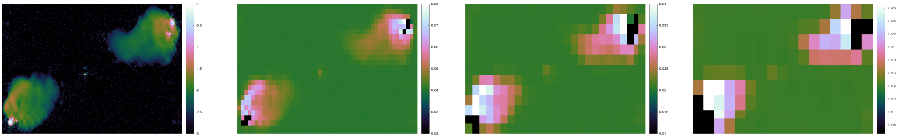

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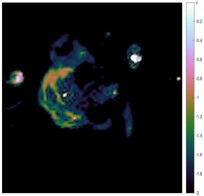


(a) Recovered image

Figure: Supernova remnant W28

# Bayesian credible regions

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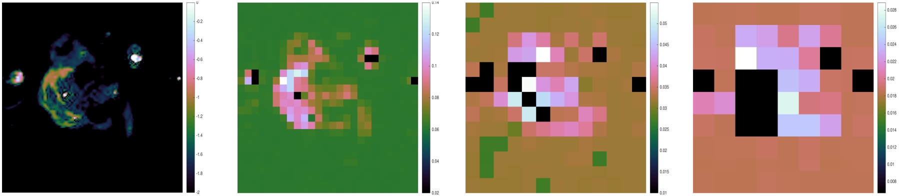

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Figure: Supernova remnant W28

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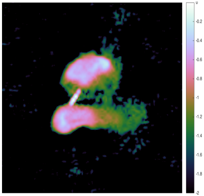


(a) Recovered image

Figure: 3C288

# Bayesian credible regions

## Preliminary results on simulations

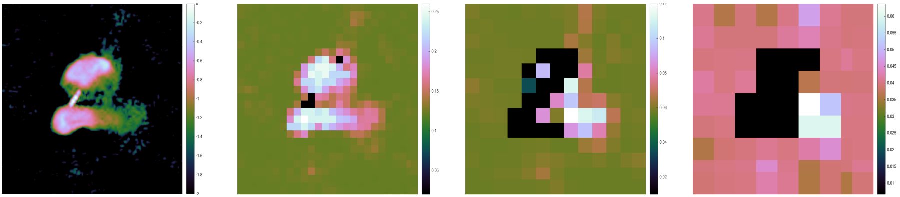

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Figure: 3C288

# Hypothesis testing

## Method

- Is structure in an image physical or an artefact?
- Can we make precise statistical statements?
- Perform hypothesis tests using Bayesian credible regions (Pereyra 2016b).

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### Hypothesis testing of physical structure

1. Cut out region containing structure of interest from recovered image $x_{\text{map}}$.
2. Inpaint background (noise) into region, yielding surrogate image $x'_{\text{map}}$.
3. Test whether $x'_{\text{map}} \in C_\alpha$:
   - If $x'_{\text{map}} \notin C_\alpha$ then reject hypothesis that structure is an artefact with confidence $(1-\alpha)\%$, *i.e.* structure most likely physical.
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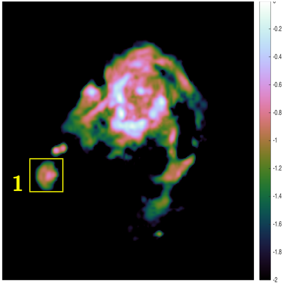


(a) Recovered image

Figure: HII region of M31

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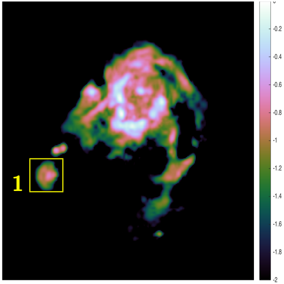


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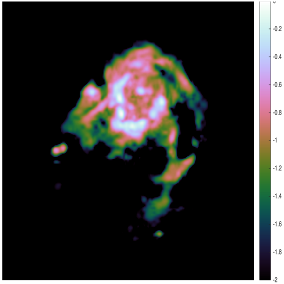


(b) Surrogate with region removed

Figure: HII region of M31

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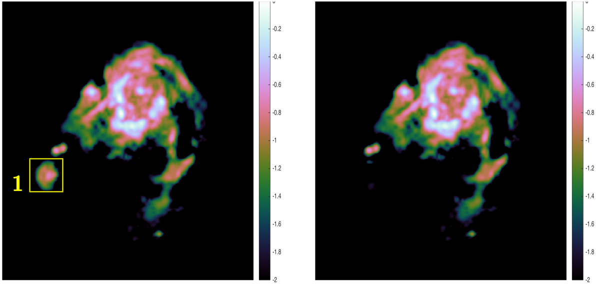


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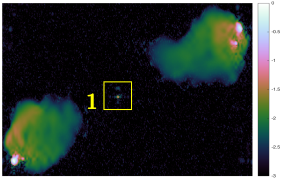
Figure: HII region of M31

Reject null hypothesis

⇒ structure physical

# Hypothesis testing

## Preliminary results on simulations

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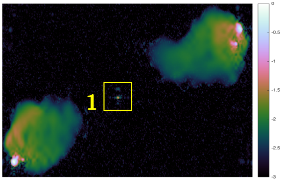
## Preliminary results on simulations

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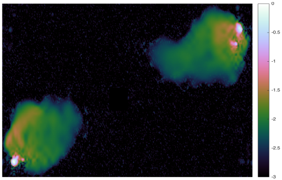


(b) Surrogate with region removed

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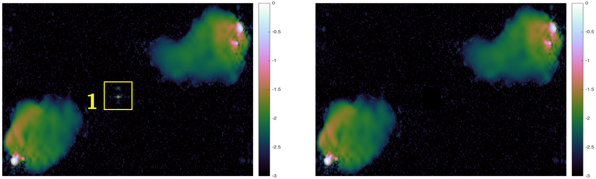


(a) Recovered image

(b) Surrogate with region removed

Figure: Cygnus A

Cannot reject null hypothesis

$\Rightarrow$ cannot make strong statistical statement about origin of structure

# Hypothesis testing

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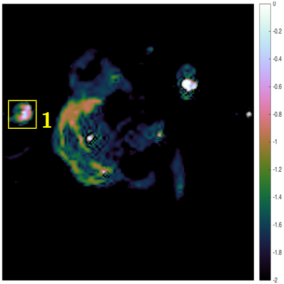


(a) Recovered image

Figure: Supernova remnant W28

# Hypothesis testing

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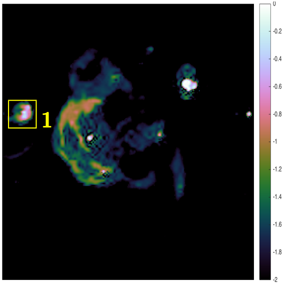


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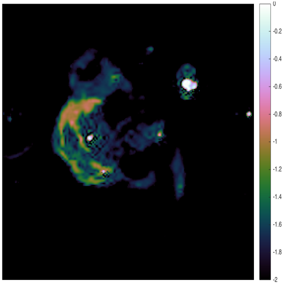


(b) Surrogate with region removed

Figure: Supernova remnant W28

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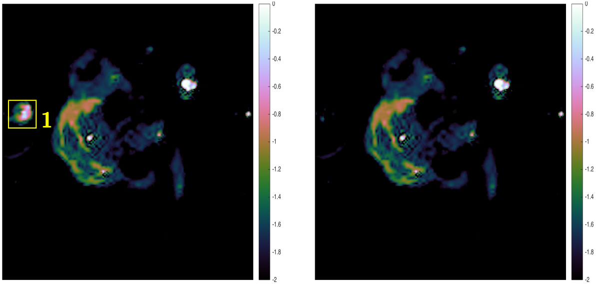


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Figure: Supernova remnant W28

Reject null hypothesis

$\Rightarrow$ structure physical

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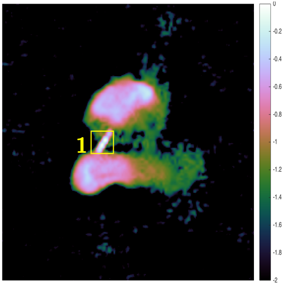
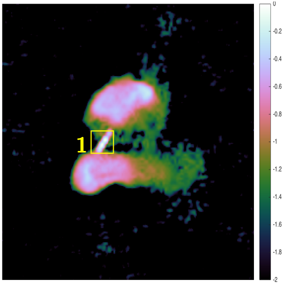



(a) Recovered image

Figure: 3C288

# Hypothesis testing

## Preliminary results on simulations

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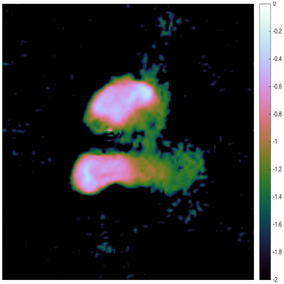


(b) Surrogate with region removed

Figure: 3C288

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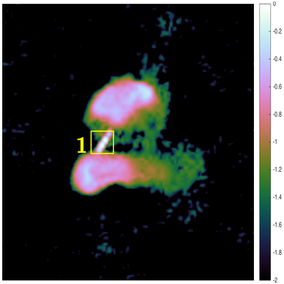


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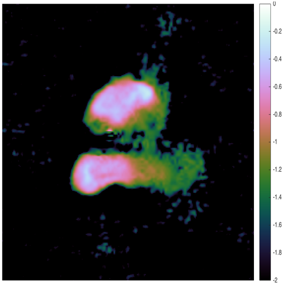


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# Conclusions

- Sparse priors (*cf.* compressive sensing) shown to be highly effective (see talks by Luke Pratley, Alex Onose, Vijay Kartik).
- Also seek statistical interpretation to recover error information.
- Proximal MCMC sampling can support sparse priors in full statistical framework (P-MALA).
- Combine error estimation with fast sparse regularisation:
  - Recover Bayesian credible regions.
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