Spin scale-discretised wavelets on the sphere for the analysis of CMB polarisation

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CMB polarisation

- CMB polarisation contains a wealth of cosmological information; but extracting this information is challenging.
- Evidence for primordial gravitational waves from BICEP2?
- Observe Q and U Stokes parameters.
- Construct $Q \pm iU$, which is a spin ± 2 field: $(Q \pm iU)'(\omega) = e^{\mp i2\chi}(Q \pm iU)(\omega)$.
- Extract and analyse cosmological **maps**.

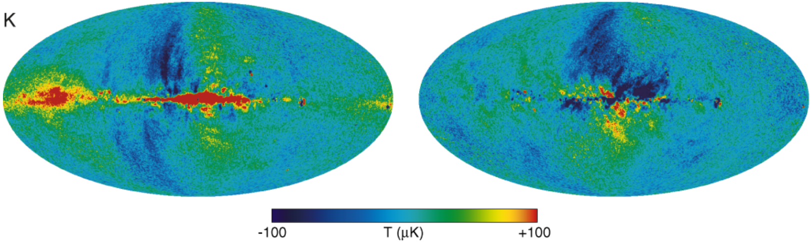

Figure: WMAP K-band Q and U maps.

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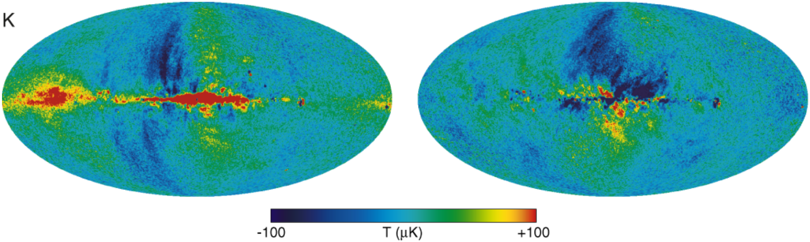

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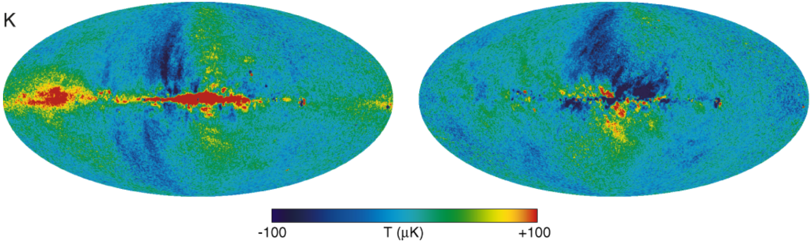

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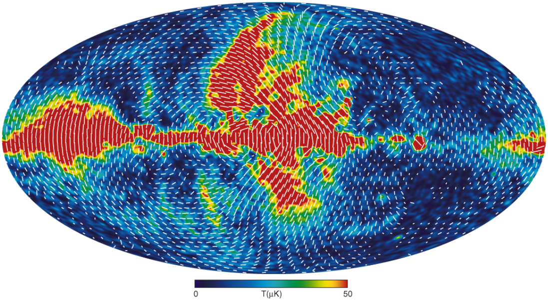

Figure: WMAP K-band $Q + \mathrm{i}U$ map.

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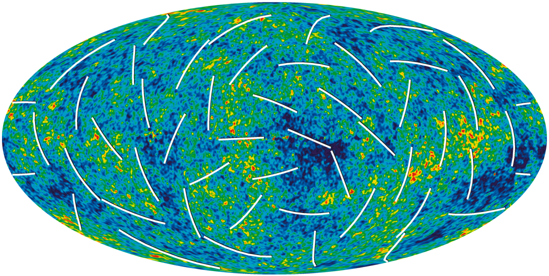

Figure: WMAP cosmological temperature and polarisation.

Outline

1. Spin scale-discretised wavelets on the sphere
2. Fast algorithms
3. E/B separation

Recall wavelet transform in Euclidean space

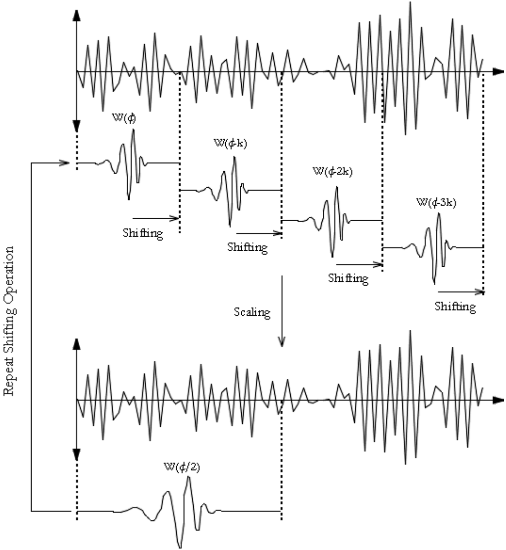

Figure: Wavelet scaling and shifting [Credit: http://www.wavelet.org/tutorial/]

Wavelets on the sphere

Dilation and translation

- Construct wavelet atoms from affine transformations (dilation, translation) on the sphere of a mother wavelet.
- The natural extension of translations to the sphere are rotations. Rotation of a function f on the sphere is defined by

$$[\mathcal{R}(\rho)f](\omega) = f(\mathsf{R}_\rho^{-1} \cdot \omega), \quad \omega = (\theta, \varphi) \in \mathbb{S}^2, \quad \rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3) \ .$$

translation

- How define dilation on the sphere?
  - Stereographic projection
    Antoine & Vandergheynst (1999), Wiaux *et al.* (2005)
  - Harmonic dilation wavelets
    McEwen *et al.* (2006), Sanz *et al.* (2006)
  - Isotropic undecimated wavelets
    Starck *et al.* (2005), Starck *et al.* (2009)
  - Needlets
    Narcowich *et al.* (2006), Baldi *et al.* (2009), Marinucci *et al.* (2008), Geller *et al.* (2008)
  - Scale-discretised wavelets
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Scale-discretised wavelets on the sphere

Wavelet construction

- *Exact reconstruction with directional wavelets on the sphere*
  Wiaux, McEwen, Vandergheynst, Blanc (2008)

- Extend to spin functions.

- Scale-discretised wavelet ${}_s\Psi^j \in \mathrm{L}^2(\mathbb{S}^2)$ defined in harmonic space:

$$ {}_s\Psi^j_{\ell m} \equiv \kappa^j(\ell) s_{\ell m} \: . $$

- Admissible wavelets constructed to satisfy a resolution of the identity:

$$ \underbrace{|{}_s\Phi_{\ell 0}|^2}_{\text{scaling function}} + \sum_{j=0}^{J} \sum_{m=-\ell}^{\ell} \underbrace{|{}_s\Psi^j_{\ell m}|^2}_{\text{wavelet}} = 1 \, , \quad \forall \ell . $$

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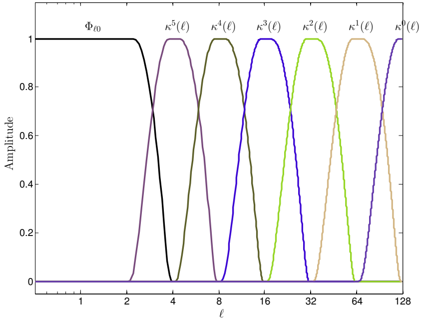

Figure: Harmonic tiling on the sphere.

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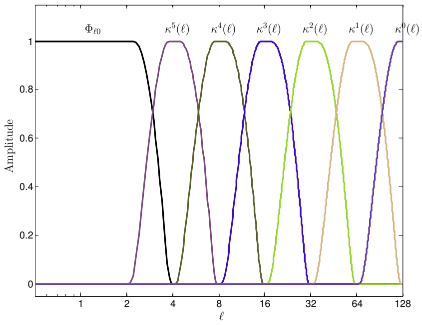

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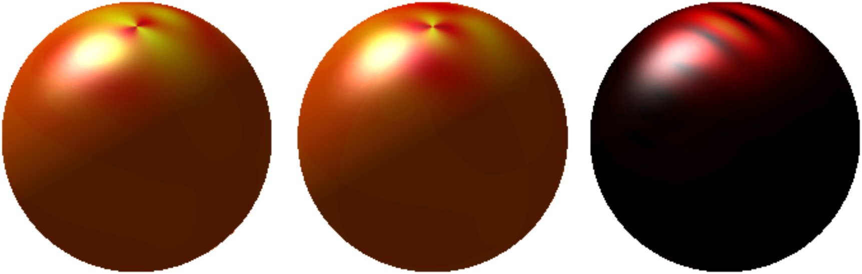

(a) Real$({}_s\Psi^j)$ (b) Imag$({}_s\Psi^j)$ (c) Abs$({}_s\Psi^j)$

Figure: Spin scale-discretised wavelets on the sphere.

Scale-discretised wavelets on the sphere

Forward and inverse transform (i.e. analysis and synthesis)

- The spin scale-discretised wavelet transform is given by the usual projection onto each wavelet:

$$\underbrace{W^{{}_s\Psi^j}(\rho) = \langle {}_sf,\, \mathcal{R}_\rho\, {}_s\Psi^j \rangle}_{\text{projection}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)\, {}_sf(\omega) (\mathcal{R}_\rho\, {}_s\Psi^j)^*(\omega) \; .$$

- Wavelet coefficients are scalar and not spin.
- Wavelet coefficients live in $\mathrm{SO}(3) \times \mathbb{Z}$; thus, directional structure is naturally incorporated.

- The original function may be recovered exactly in practice from the wavelet (and scaling) coefficients:

$${}_sf(\omega) = \underbrace{\sum_{j=0}^{J}}_{\text{finite sum}} \underbrace{\int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{{}_s\Psi^j}(\rho) (\mathcal{R}_\rho\, {}_s\Psi^j)(\omega) \; .}_{\text{wavelet contribution}}$$

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finite sum

wavelet contribution

Scale-discretised wavelets on the sphere
Steerability

- By imposing an azimuthal band-limit N, we recover steerable wavelets.
- By the linearity of the wavelet transform, steerability extends to wavelet coefficients:

$$W^{{}_s\Psi^j}(\alpha, \beta, \gamma) = \sum_{g=0}^{M-1} z(\gamma - \gamma_g)\, W^{{}_s\Psi^j}(\alpha, \beta, \gamma_g) \,.$$

steerability

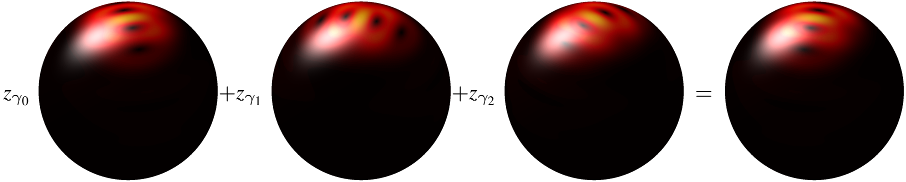

Figure: Steered wavelet computed from basis wavelets.

Fast Wigner transform

Fourier transform on the rotation group $\mathrm{SO}(3)$

- Wavelet coefficients live on the rotation group $\mathrm{SO}(3)$: $W^{s\Psi^j} \in \mathrm{L}^2(\mathrm{SO}(3))$.
- Develop fast wavelet transforms by considering their (Wigner) harmonic representation.
- Signal on the rotation group $F \in \mathrm{L}^2(\mathrm{SO}(3))$ may expressed by Wigner decomposition:

$$F(\rho) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \frac{2\ell+1}{8\pi^2} F^{\ell}_{mn} D^{\ell *}_{mn}(\rho)$$

where Wigner coefficients given by usual projection onto basis functions:

$$F^{\ell}_{mn} = \langle F, D^{\ell *}_{mn} \rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) F(\rho)\, D^{\ell}_{mn}(\rho) \, .$$

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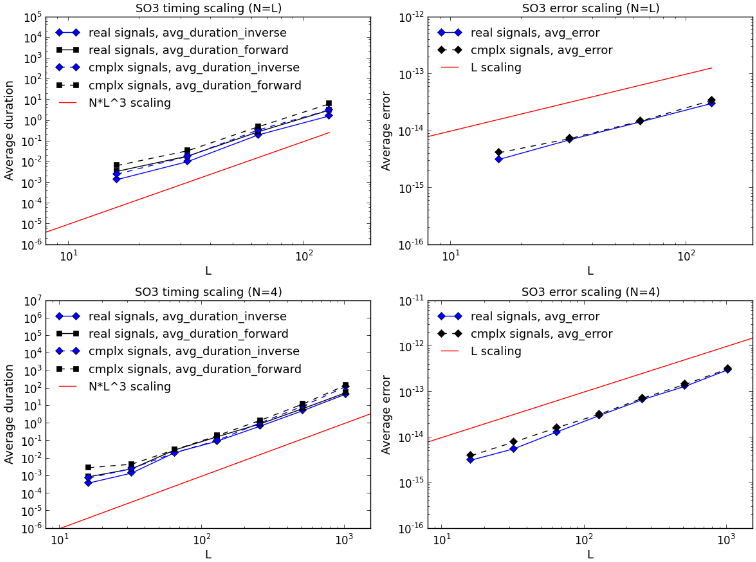
Fast Wigner transform

Timing and accuracy



Fast spin scale-discretised wavelet transform

Exact and efficient computation via Wigner transforms

- Wavelet analysis can be posed as an inverse Wigner transform on $\mathrm{SO}(3)$:

$$\left(W^{\Psi^j}\right)^{\ell}_{mn} = \frac{8\pi^2}{2\ell+1} f_{\ell m} \, \Psi^{j*}_{\ell n} \, .$$

analysis

- Wavelet synthesis can be posed as a forward Wigner transform on $\mathrm{SO}(3)$:

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synthesis

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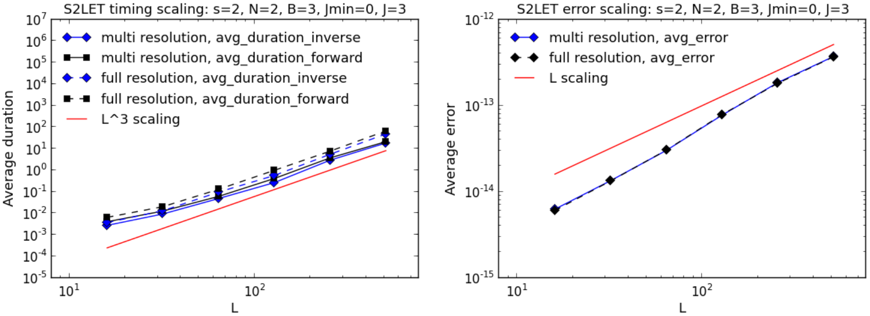

E/B separation

From $Q \pm \mathrm{i}U$ to E and B maps

- Different physical processes exhibit different symmetries and thus behave differently under parity transformation.
- Can exploit this property to separate signals arising from different underlying physical mechanisms.
- Decompose $Q \pm \mathrm{i}U$ into parity even and odd components:

$$\tilde{E}(\omega) = -\frac{1}{2}\left[\bar{\eth}^2 (Q + \mathrm{i}U)(\omega) + \eth^2 (Q - \mathrm{i}U)(\omega)\right]$$

E-mode

$$\tilde{B}(\omega) = \frac{\mathrm{i}}{2}\left[\bar{\eth}^2 (Q + \mathrm{i}U)(\omega) - \eth^2 (Q - \mathrm{i}U)(\omega)\right]$$

B-mode

where $\bar{\eth}$ and $\eth$ are spin lowering and raising operators, respectively.

- Number of existing techniques:
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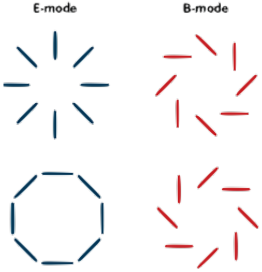

Figure: E-mode (even parity) and B-mode (odd parity) signals [Credit: `http://www.skyandtelescope.com/`].

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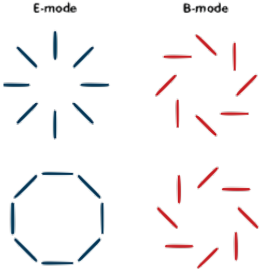

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E/B separation

Spin and scalar scale-discretised wavelets

- Spin wavelet transform of $Q + \mathrm{i}U$ (observable):

$$\boxed{W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho) = \langle Q + \mathrm{i}U,\ \mathcal{R}_\rho\, {}_2\Psi^j \rangle} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)(Q + \mathrm{i}U)(\omega)(\mathcal{R}_\rho\, {}_2\Psi^j)^*(\omega)\ .$$

spin wavelet transform

- Scalar wavelet transforms of E and B (non-observable):

$$\boxed{W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = \langle \tilde{E},\ \mathcal{R}_\rho\, {}_0\tilde{\Psi}^j \rangle} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)\tilde{E}(\omega)(\mathcal{R}_\rho\, {}_0\tilde{\Psi}^j)^*(\omega)\ ,$$

scalar wavelet transform

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scalar wavelet transform

where ${}_0\tilde{\Psi}^j \equiv \bar{\eth}^2\, {}_2\Psi^j$.

- Spin wavelet coefficients of $Q + \mathrm{i}U$ are connected to scalar wavelet coefficients of E/B:

$$W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = -\mathrm{Re}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \quad \text{and} \quad W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = -\mathrm{Im}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right]\ .$$

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$$W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = -\mathrm{Re}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \quad \text{and} \quad W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = -\mathrm{Im}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] .$$

E/B separation

Spin and scalar scale-discretised wavelets

- Spin wavelet transform of $Q + \mathrm{i}U$ (observable):

$$\boxed{W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho) = \langle Q + \mathrm{i}U,\ \mathcal{R}_\rho\, {}_2\Psi^j \rangle} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) (Q + \mathrm{i}U)(\omega) (\mathcal{R}_\rho\, {}_2\Psi^j)^*(\omega) \ .$$

spin wavelet transform

- Scalar wavelet transforms of E and B (non-observable):

$$\boxed{W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = \langle \tilde{E},\ \mathcal{R}_\rho\, {}_0\tilde{\Psi}^j \rangle} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \tilde{E}(\omega) (\mathcal{R}_\rho\, {}_0\tilde{\Psi}^j)^*(\omega) \ ,$$

scalar wavelet transform

$$\boxed{W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = \langle \tilde{B},\ \mathcal{R}_\rho\, {}_0\tilde{\Psi}^j \rangle} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \tilde{B}(\omega) (\mathcal{R}_\rho\, {}_0\tilde{\Psi}^j)^*(\omega) \ ,$$

scalar wavelet transform

where ${}_0\tilde{\Psi}^j \equiv \bar{\eth}^2\, {}_2\Psi^j$.

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E/B separation

Using scale-discretised wavelets

Algorithm to recover E/B signals using scale-discretised wavelets

1. Compute spin wavelet transform of $Q + \mathrm{i}U$:

$$(Q + \mathrm{i}U)(\omega) \xrightarrow[\text{S2LET}]{\text{Spin wavelet transform}} W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)$$

2. Account for mask in harmonic and spatial domains simultaneously:

$$W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho) \xrightarrow{\text{Mitigate mask}} \widehat{W}^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)$$

3. Construct E/B maps:

(a) $$W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = -\mathrm{Re}\left[\widehat{W}^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \xrightarrow[\text{S2LET}]{\text{Inverse scalar wavelet transform}} \tilde{E}(\omega)$$

(b) $$W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = -\mathrm{Im}\left[\widehat{W}^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \xrightarrow[\text{S2LET}]{\text{Inverse scalar wavelet transform}} \tilde{B}(\omega)$$

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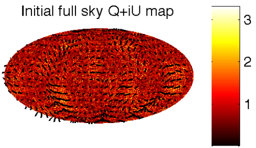
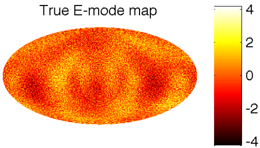
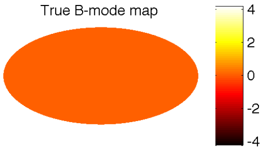
E/B separation

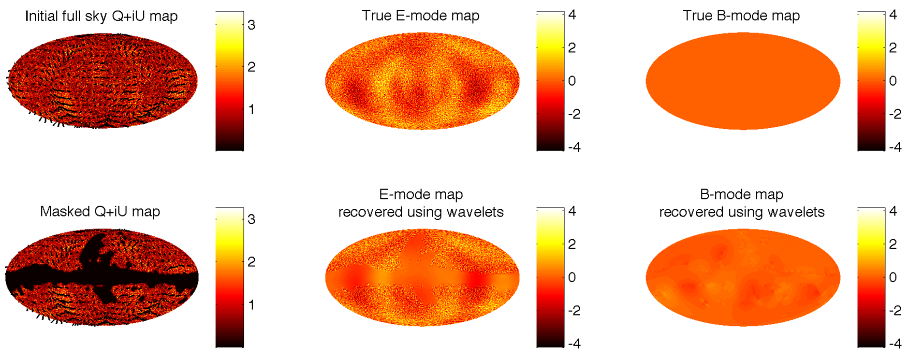
Simulations



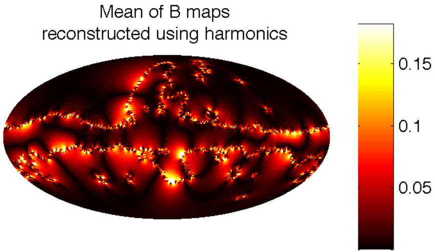


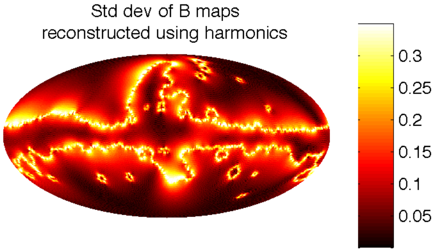


E/B separation
Simulations



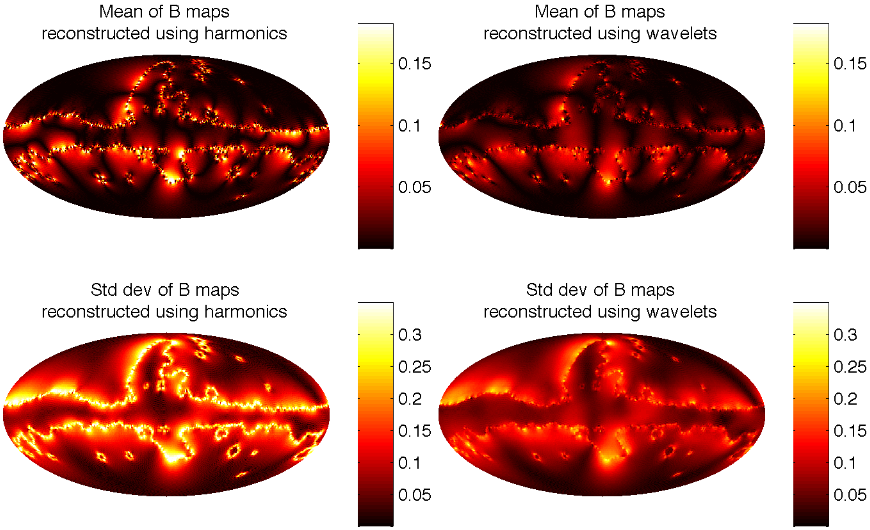
E/B separation
Preliminary results





E/B separation

Preliminary results

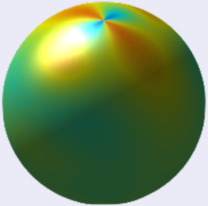


Summary

Spin scale-discretised wavelets are a powerful tool to study CMB polarisation, with an elegant and practical connection to scalar wavelet transforms of E/B maps.

S2LET code

http://www.s2let.org



S2LET: A code to perform fast wavelet analysis on the sphere

Leistedt, McEwen, Vandergheynst, Wiaux (2012)

- C, Matlab, IDL, Java
- Supports only axisymmetric wavelets at present
- Extensions for directional and steerable wavelets, faster algos, and spin wavelets coming very soon